Contests with Interpersonal Projection*

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Abstract

Gagnon-Bartsch, Pagnozzi, and Rosato (2021) study projection bias in auctions in which only the winner pays. Motivated by unconditional investments in real-world competitions, we introduce a loser-pay component and analyze the resulting contests under projection bias. Assuming rationality, the loser-pay fraction θ—interpreted as required preliminary work—reduces the expected winner's effort (which corresponds to the completed project's quality) under both first- and second-price winner-pay rules. However, we show that under the second-price rule, projection bias can cause the expected winner's effort to initially increase with θ. Conversely, under the first-price rule, the expected winner's effort monotonically decreases in θ despite the presence of projection bias. Nevertheless, first-price contests consistently yield higher expected winner's effort than second-price contests for any given θ and degree of projection bias. We also provide comparative statics and cross-format comparisons for the expected total effort.

Keywords: contests; procurement; projection bias; total effort; winner's effort; hybrid payment rule

JEL Classification Codes: C72; D44; D81; D91

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1 Introduction

People tend to overestimate the similarity between their own preferences and those of others, a phenomenon termed *interpersonal projection bias* (or the false-consensus effect). This bias is well documented in psychology (Ross et al., 1977; Marks and Miller, 1987; Mullen et al., 1985) and behavioral economics (Ambuehl et al., 2021; Bushong and Gagnon-Bartsch, 2024), and manifests across various domains, including political preferences (Koestner et al., 1995; Morwitz and Pluzinski, 1996; Delavande and Manski, 2012); risk attitudes (Faro and Rottenstreich, 2006; Bollen and Posavac, 2018); and consumer decision-making (Orhun and Urminsky, 2013).

The seminal work of Gagnon-Bartsch, Pagnozzi, and Rosato (2021, henceforth GPR) studies the effects of projection bias on equilibrium bidding and allocation in standard winner-pay auctions. However, in many real-world competitions, *all* contenders make irrevocable investments that are unconditional on winning. For example, in public procurement, bidders must often submit prototypes, samples, or demonstration models before their bids can be evaluated or contracted upon. These pre-award investments represent sunk costs for all bidders, whereas only the winner is selected to execute the remaining tasks specified in the bid.[1] Similarly, in grant-making, applicants must exert sunk effort to prepare proposals before a research project can be implemented.

Motivated by the prevalence of sunk investments, we introduce a loser-pay fraction to the GPR framework and analyze the resulting contests. Specifically, losers pay a fraction $\theta \in [0,1]$ of their bids, which captures the cost of pre-award preliminary work. For the winner's payment, following GPR, we examine both the first-price format—in which the winner is obligated to fulfill the submitted bid—and the second-price format—in which the winner need only match the highest losing bid.[2] The latter rule is particularly relevant when the progress of preliminary work is transparent to bidders and can be dynamically adjusted. Consistent with our motivating examples, we interpret a bid as the promised quality of the project upon completion. Accordingly, we refer to the incurred costs—i.e., the cost of sunk preliminary work for the losing bidders and the cost to achieve the promised quality or the highest losing promised quality for the winner—as effort.

The contest framework allows us to address several issues absent from the standard

[1]For instance, in New York City's two-year Taxi of Tomorrow competition starting from late 2009, Nissan, Ford, and Karsan incurred engineering, design, and proposal-preparation costs to tailor their vehicles to the city's requirements before Nissan was awarded the 10-year supply contract in 2011.

[2]This parameterization nests standard auction formats as special cases: $\theta = 0$ recovers standard first- and second-price auctions; $\theta = 1$ corresponds to all-pay contests, in which the first-price format becomes the standard all-pay auction and the second-price format reduces to a war of attrition, as in Krishna and Morgan (1997).

auction setting. First, it naturally raises the question of how the preliminary-work fraction θ affects equilibrium outcomes. Second, in this setting, the winner's effort emerges as a distinct and crucial performance measure. In the winner-pay auction setting, total effort and the winner's effort coincide because losers incur no costs. However, with a positive loser-pay fraction, these two measures diverge. The winner's effort warrants independent analysis because it determines the quality of the top-performing entry—a metric of primary interest in applications whereby only the winning submission is ultimately implemented, such as public procurement, grant-making, patent races, and crowdsourcing contests.[3] Finally, whereas GPR show that first-price auctions outperform second-price auctions in revenue with projection-biased bidders, it remains an open question how the two winner-pay rules compare regarding total effort (the counterpart of revenue) and the winner's effort in the presence of a loser-pay component.

This paper addresses these questions and explores their implications for contest design. To facilitate comparison with existing revenue results in the auction literature, we first analyze the expected total effort. Absent projection bias, standard revenue equivalence implies that the preliminary-work requirement or loser-pay fraction θ has no impact on expected total effort. With projection bias, we show that this irrelevance still holds for the first-price rule; for the second-price rule, however, expected total effort strictly increases in θ (Proposition 1). This result highlights the interplay between projection bias and the winner's payment format. Under the first-price rule, the downward adjustment in bids exactly offsets the additional effort extracted from losers. Under the second-price format, however, projection bias amplifies the perceived incentive to secure a win; furthermore, because the winner does not pay their own bid, aggressive bidding is less costly conditional on winning. Consequently, equilibrium bids do not adjust downward sufficiently to offset the extra loser-pay effort, which causes the expected total effort to increase in θ. Regarding cross-format comparisons, we establish that GPR's ranking—whereby the first-price format yields higher expected total effort than the second-price format—robustly extends to any loser-pay fraction $\theta \in [0, 1]$ (Proposition 2).

We then examine the expected winner's effort. Under the first-price rule, the expected winner's effort unambiguously decreases with the loser-pay fraction θ, with or without projection bias. In contrast, under the second-price rule, while the expected winner's effort decreases with θ for the rational case, projection bias renders this monotonicity indeterminate (Proposition 3). Specifically, with significant projection bias, the winner's

[3]Reflecting its practical significance, the winner's effort has received increasing attention in recent contest literature (e.g., Moldovanu and Sela, 2006; Serena, 2017; Fu and Wu, 2022; Wasser and Zhang, 2023; Barbieri and Serena, 2021, 2024).

effort under the second-price rule can initially increase with θ provided that values are sufficiently dispersed or the number of bidders is small. Despite the nuance, we establish that the expected winner's effort under the first-price rule is always greater than that under the second-price rule for any given level of projection bias and loser-pay fraction (Proposition 4).

Our main results yield the following implications for contest design. To the extent that the procurement agency can control the preliminary work fraction θ, our results shed light on when (not) to require more preliminary or demonstrative work. While increasing this requirement is never optimal under rationality or the first-price rule, a procurement agency seeking to maximize the expected total effort under the second-price rule (i.e., when renegotiation is unavoidable) should require the project to be completed entirely upfront. Furthermore, if the objective is to maximize the expected winner's effort, it can be optimal to mandate a certain fraction of pre-award work under the second-price rule. Finally, our finding that the first-price rule dominates the second-price rule for both objectives—across any degree of projection bias and loser-pay fraction—suggests that the procurement agency must prevent information leakage regarding the progress of competitors' preliminary work.

Related literature Our study joins the burgeoning theoretical literature on interpersonal projection bias. Prior research has examined this bias in various contexts, such as voting in elections (Goeree and Grosser, 2007) and bilateral bargaining (Madarász, 2021).[4] As discussed above, our paper is most closely related to GPR, who introduce taste projection in auctions.[5] Subsequent work has extended this framework to dynamic pricing (Gagnon-Bartsch and Rosato, 2024); public good provision (Barbieri and Serena, 2025); and supplier production contracts (Morita, Muramoto, and Sogo, 2025). We contribute to this literature by introducing a loser-pay component to the GPR framework to capture unconditional investments in competitions. This enables us to study the impact of the preliminary-work requirement and how it is affected by projection bias across different winner-pay rules.

Our contest framework also connects to the literature on unconditional investments and hybrid payment rules. Siegel (2010) develops a general framework under complete information that accommodates both unconditional and winning-contingent investment costs. As an application, this study analyzes the same first-price parameterization as ours: The winner bears the full cost of the submitted bid, whereas each loser bears only a fraction

[4]Madarász (2012, 2021) considers information projection, whereby individuals overestimate the extent to which their information is shared by others.

[5]For a different approach to modeling projection bias in auctions, see Breitmoser (2019).

of their bid.[6] In an extension of their incomplete-information contest with sequential entry, Deng, Fu, Wu, and Zhu (2024) adopt the same first-price parameterization; however, their analysis focuses on the later-mover advantage. We contribute to the literature by incorporating projection bias, comparing first- and second-price rules, and examining the expected winner's effort as a distinct performance measure.

This paper also complements the literature on contests with behavioral elements. Prior empirical and experimental studies document systematic overbidding (e.g., Gneezy and Smorodinsky, 2006; Lugovskyy et al., 2010; Abbink et al., 2010), a phenomenon variously attributed to overconfidence (Santos-Pinto and Sekeris, 2022); bounded rationality (Sheremeta, 2011); the non-monetary utility of winning (Sheremeta, 2010); and other-regarding preferences (Balafoutas et al., 2012).[7] We show that projection bias provides an additional rationale for aggressive bidding in general contest settings.

The remainder of the paper proceeds as follows. Section 2 sets up the model, characterizes the equilibrium, and lays out the rational benchmark. Section 3 presents comparative statics for total effort with respect to the loser-pay fraction and compares total effort across the first- and second-price winner-pay rules. Section 4 analyzes the expected winner's effort. Section 5 presents numerical exercises to quantify the main effects across various value distributions and numbers of contestants, and Section 6 concludes. Appendix A contains the proofs of all formal results, and Appendix B provides supplementary numerical simulations.

2 Model and Preliminaries

In this section, we introduce the model, characterize the equilibrium, and establish the rational benchmark.

2.1 Model Setup

Consider a procurement in which $n \geq 2$ suppliers compete for a contract. Each supplier i submits a proposal that specifies the quality of work q_i. We assume that the cost of delivering a project of quality q is q.[8] To credibly promise a quality q, a fraction $\theta \in [0, 1]$ of the project (e.g., a prototype) must be completed upfront, incurring a cost of

[6] The two-player complete-information rank-order framework of Baye, Kovenock, and De Vries (2012) likewise nests this parameterization as a special case.

[7] For comprehensive surveys, see Sheremeta (2013) and Dechenaux et al. (2015).

[8] Because q represents a quality score, this linear cost assumption is a standard normalization. For any strictly increasing cost function $c(q)$, one can redefine the quality score as $\widehat{q} := c(q)$. In terms of this rescaled quality score, the cost of delivering $\widehat{q}$ is simply $\widehat{q}$.

θq. Alternatively, θq can be interpreted as the minimum amount of preliminary work required before the proposal's quality can be assessed.

Upon completion of the preliminary work, the contract is awarded to the supplier who submits the highest-quality bid q_i, yielding a private payoff of v_i.[9,10] This parameter v_i represents supplier i's private valuation for the contract. The valuations are independently and identically distributed according to a cumulative distribution function (CDF) $F : \mathcal{V} \to [0, 1]$, where $\mathcal{V} := [\underline{v}, \bar{v}]$ with $\underline{v} \geq 0$. We assume that F admits a continuous probability density function $f = F'$, and $f(v) > 0$ for all $v \in [\underline{v}, \bar{v}]$.

We consider two procurement formats, which correspond to two winner-pay rules in standard auctions. Under the first-price rule, the winner is obligated to deliver the project at the promised quality level; under the second-price rule, the winner need only deliver a quality level matching the runner-up's promise. Analogous to the relationship between second-price and English auctions, the second-price format can be understood as a reduced-form model for settings in which the progress of the preliminary work is transparent among suppliers.[11]

The procurement setting with unconditional investments admits a general contest interpretation, which nests standard winner-pay auctions when $\theta = 0$ and all-pay contests when $\theta = 1$. Following the contest literature, we refer to suppliers' costly investments as effort.

Following GPR, we model interpersonal taste projection by assuming that each contestant perceives the distribution of others' private values as overly centered around their own preferences. More formally, contestant i with valuation v_i believes that the valuation of any other contestant j is given by

$$\widehat{v}_j(v_i) = \alpha v_i + (1 - \alpha) v_j,$$

[9]As discussed later in Remark 1, this model can be reinterpreted as a contest featuring a common prize but heterogeneous cost efficiencies. We adopt the independent private value formulation to facilitate comparison with GPR's baseline auction setting.

[10]The winner-selection mechanism is modeled as deterministic rather than probabilistic for two reasons. First, deterministically awarding the contract to the highest-ranked bid is a standard assumption in the procurement literature (see, e.g., Thiel, 1988; Che, 1993; Branco, 1997; Asker and Cantillon, 2008, 2010). This approach facilitates direct comparisons with established results from the auction literature. Second, the second-price format lacks a straightforward counterpart under noisy winner-selection mechanisms such as Tullock contests. Under stochastic selection, the runner-up's bid does not naturally dictate the winner's performance obligation. Formulating such a format would require a framework that specifies how noisy evaluations map bids into implemented quality, which would substantially complicate the analysis.

[11]Specifically, the outcome under the second-price rule is equivalent to the following dynamic game: A public quality clock q rises continuously from zero. To remain active at level q, a supplier must have completed preliminary work amounting to θq. The clock stops when only one supplier remains active.

where $\alpha \in [0, 1)$ measures the intensity of projection bias.[12] Consistent with GPR, this belief structure extends to higher-order beliefs: Contestant i believes that it is common knowledge that all contestants share their perception of the value distribution. Let $\widehat{F}(\cdot|v_i)$ denote the CDF of $\widehat{v}_j(v_i)$. Consequently, contestant i with valuation v_i erroneously believes that each opponent's valuation is represented by the following random variable:

$$\widehat{V}(v_i) = \alpha v_i + (1 - \alpha)V,$$

where $V \sim F$ is the true random variable that describes each contestant's valuation. Simple algebra verifies that

$$\widehat{V}(v_i) \leq v \Leftrightarrow \alpha v_i + (1 - \alpha)V \leq v \Leftrightarrow V \leq \frac{v - \alpha v_i}{1 - \alpha}.$$

The perceived CDF $\widehat{F}(v|v_i)$ can then be derived as

$$\widehat{F}(v|v_i) = \mathbb{P}\left(\widehat{V}(v_i) \leq v\right) = \begin{cases} 0 & \text{if } v < \underline{v}(v_i), \\ F\left(\frac{v-\alpha v_i}{1-\alpha}\right) & \text{if } v \in \mathcal{V}(v_i), \\ 1 & \text{if } v > \bar{v}(v_i). \end{cases} \tag{1}$$

Here, $\mathcal{V}(v_i) := [\underline{v}(v_i), \bar{v}(v_i)]$ denotes the support of contestant i's perceived valuation distribution, with its lower and upper bounds given by $\underline{v}(v_i) := \alpha v_i + (1 - \alpha)\underline{v}$ and $\bar{v}(v_i) := \alpha v_i + (1 - \alpha)\bar{v}$, respectively.

Remark 1 The baseline model is isomorphic to a setting in which heterogeneity resides on the cost side. Suppose all suppliers compete for a common contract value $W > 0$ and each supplier i has a private ability $a_i \sim F$ on the support $[\underline{a}, \bar{a}]$ with $\underline{a} > 0$, such that producing quality q costs q/a_i. Scaling each supplier i's payoff by a_i and defining $v_i := a_i W$ recovers the original payoff structure. Moreover, ability projection—in which a contestant perceives opponent j's ability as $\hat{a}_j = \alpha a_i + (1 - \alpha)a_j$—maps directly to value projection $\hat{v}_j = \alpha v_i + (1-\alpha)v_j$, because $\hat{v}_j = W\hat{a}_j$. Consequently, all equilibrium strategies and effort measures are preserved. This equivalence implies that the projection bias in our setting can be naturally reinterpreted as contestants overestimating the similarity between their own capabilities and those of their opponents. To facilitate direct comparison with GPR, we maintain the value-based formulation throughout.

[12]In the extreme case in which $\alpha = 1$, every contestant believes that all opponents share their exact valuation, reducing the game to a complete-information contest. As long as $\theta > 0$, the equilibrium strategy in this scenario becomes mixed, leading to a discontinuity in the expected total effort and the expected winner's effort at $\alpha = 1$. As such, we restrict our analysis to $\alpha < 1$.

2.2 Equilibrium Characterization

We follow GPR to adopt *naïve Bayesian equilibrium* (NBE) as the solution concept and focus on the symmetric equilibrium in which all contestants use the same strategy. Let $\Gamma(F)$ denote the contest game with rational contestants and value distribution $F(\cdot)$. A projection-biased contestant i with valuation v_i believes they are participating in the contest $\Gamma\Big(\widehat{F}(\cdot|v_i)\Big)$. Let $\widetilde{\beta}(\cdot|v)$ denote a rational contestant's strategy in the symmetric Bayesian Nash equilibrium (BNE) of this perceived contest $\Gamma\Big(\widehat{F}(\cdot|v)\Big)$. An NBE is formally defined as follows.

Definition 1 ***(Naïve Bayesian Equilibrium)*** *All contestants using the same strategy $\widehat{\beta}(\cdot)$ constitutes a symmetric naïve Bayesian equilibrium of the contest game with projection-biased contestants if $\widehat{\beta}(v) = \widetilde{\beta}(v|v)$ for all $v \in \mathcal{V}$.*

Recall that $\widetilde{\beta}(v'|v)$ represents a type-v contestant's perceived bidding function (their belief about what a contestant with valuation v' would bid), while $\widehat{\beta}(v) = \widetilde{\beta}(v|v)$ gives the actual bid of a contestant with valuation v. By definition, to solve the symmetric NBE of the contest game, it suffices to derive the symmetric BNE of the *perceived* contest for each valuation v.[13]

For notational convenience, denote $H(v) = F^{n-1}(v)$. Further, we use superscripts I and II to indicate first- and second-price contests, respectively. The following result ensues.

Lemma 1 ***(Equilibrium Characterization)*** *Fixing $\alpha \in [0,1)$ and $\theta \in [0,1]$, the unique symmetric NBE bidding strategies in first- and second-price contests, denoted by $\widehat{\beta}^I(v)$ and $\widehat{\beta}^{II}(v)$, respectively, are:*[14]

$$\widehat{\beta}^I(v) = \alpha G^I(v) + (1-\alpha)\beta^I(v), \tag{2}$$

$$\widehat{\beta}^{II}(v) = \alpha G^{II}(v) + (1-\alpha)\beta^{II}(v), \tag{3}$$

where

$$G^I(v) = \frac{vH(v)}{(1-\theta)H(v)+\theta},$$

[13] The symmetric NBE of this contest is unique. Under the distributional assumptions outlined in Section 2.1, the contest $\Gamma(F)$ admits a unique symmetric BNE. Because the perceived distribution $\widehat{F}(\cdot|v)$ satisfies these regularity conditions for all $v \in \mathcal{V}$, each perceived contest $\Gamma\Big(\widehat{F}(\cdot|v)\Big)$ similarly admits a unique symmetric BNE. This, in turn, guarantees the uniqueness of the symmetric NBE via the mapping $\widehat{\beta}(v) = \widetilde{\beta}(v|v)$.

[14] At $\theta = 0$ and $\theta = 1$, the expressions below are defined via their respective one-sided limits from the interior $\theta \in (0,1)$. Under the distributional assumptions outlined in Section 2.1, these limits exist and are finite F-almost everywhere. For details, see the proof of Lemma 1.

$$G^{II}(v) = \frac{v}{1-\theta}\left[1 - (1 - H(v))^{\frac{1-\theta}{\theta}}\right],$$

and $\beta^I(v)$ *and* $\beta^{II}(v)$ *are the symmetric BNE strategies for rational contestants in first- and second-price contests with value distribution* $F(\cdot)$*, respectively:*

$$\beta^I(v) = \frac{\int_{\underline{v}}^{v} t dH(t)}{(1-\theta)H(v) + \theta}, \tag{4}$$

$$\beta^{II}(v) = \frac{[1-H(v)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}}^{v} \frac{t dH(t)}{[1-H(t)]^{\frac{1}{\theta}}}. \tag{5}$$

From the equilibrium characterization established in Lemma 1, we derive the effects of projection bias on equilibrium bidding:

Corollary 1 *For any* $v \in (\underline{v}, \bar{v})$*, the first-price equilibrium bid* $\widehat{\beta}^I$ *is strictly increasing in* α *for every* $\theta \in [0,1]$*. Under the second-price rule,* $\widehat{\beta}^{II}$ *is independent of* α *when* $\theta = 0$*, and is strictly increasing in* α *when* $\theta \in (0,1]$*. Consequently, for any* $\alpha \in (0,1)$*,*

$$\widehat{\beta}^I(v) > \beta^I(v) \quad \textit{for all } \theta \in [0,1],$$

whereas

$$\widehat{\beta}^{II}(v) > \beta^{II}(v) \quad \textit{for all } \theta \in (0,1].$$

Under the first-price rule, the bid inflation result is established by GPR for the winner-pay-only case (i.e., $\theta = 0$). We extend the result to any loser-pay fraction $\theta \in [0,1]$, and the intuition is similar to that in GPR. Specifically, projection-biased contestants perceive their opponents' valuations as overly concentrated around their own. Consequently, a type-v contestant perceives heightened competition and shades their bid less relative to the rational benchmark, leading to inflated equilibrium bids.

Analyzing the second-price format requires distinguishing between the winner-pay-only case and the loser-pay case. For the $\theta = 0$ case, as pointed out by GPR, projection bias has no impact on equilibrium bidding. Because the winner's cost is determined not by their own bid but rather by the highest losing bid, truthful bidding remains a weakly dominant strategy irrespective of α. By contrast, once $\theta > 0$, a contestant's payment always (at least partially) depends on their own bid. Consequently, the rationale for truthful bidding collapses, even for the rational case in which $\alpha = 0$.[15] In this environment, perceived

[15]It is straightforward to verify that truthful bidding is no longer weakly dominant in this case. For example, if all opponents bid zero, bidding marginally above zero yields a strictly higher payoff than bidding one's true valuation.

competition matters. As under the first-price rule, projection bias intensifies perceived competition, thereby inflating equilibrium bids.

Lemma 1 enables us to analyze two key performance measures of the contest. The first is *expected total effort*, which captures the aggregate investment of all contestants and parallels total revenue in auctions. This metric is of primary interest in settings such as R&D competitions, in which aggregate investment can foster the development of new technologies or industries. Denote the expected total effort under winner-pay rule $k \in \{I, II\}$ by $TE^k(\alpha, \theta)$. Straightforward calculations yield

$$TE^k(\alpha, \theta) = n \int_{\underline{v}}^{\bar{v}} e^k(v) dF(v), \tag{6}$$

where $e^k(v)$ represents the expected effort expenditure of a contestant with valuation v in contest format $k \in \{I, II\}$ and takes the form

$$e^I(v) = H(v)\widehat{\beta}^I(v) + [1 - H(v)]\, \theta\widehat{\beta}^I(v), \tag{7}$$

$$e^{II}(v) = \int_{\underline{v}}^{v} \widehat{\beta}^{II}(t) dH(t) + [1 - H(v)]\, \theta\widehat{\beta}^{II}(v). \tag{8}$$

The second is the *expected winner's effort*, which we denote by $WE^k(\alpha, \theta)$ for payment rule $k \in \{I, II\}$. In the procurement setting, this measure is particularly relevant because it corresponds to the quality of the completed project. Formally, the expected winner's effort under payment rule $k \in \{I, II\}$ can be expressed as follows:

$$WE^I(\alpha, \theta) := \int_{\underline{v}}^{\bar{v}} \widehat{\beta}^I(v) dF^n(v), \tag{9}$$

$$WE^{II}(\alpha, \theta) := \int_{\underline{v}}^{\bar{v}} \widehat{\beta}^{II}(v) dF_{(2)}(v), \tag{10}$$

where $F_{(2)}(v) := F^n(v) + nF^{n-1}(v)\Big(1 - F(v)\Big)$ is the CDF of the second-highest order statistic among $n \geq 2$ independently and identically distributed draws from $F(\cdot)$.

2.3 Rational Benchmark

To highlight how projection bias influences the comparative statics of both total effort and the winner's effort with respect to loser-pay fraction θ, as well as their respective rankings under first- versus second-price rules, we first establish benchmark results for rational contestants.

Note that total effort in our contest framework mirrors total revenue in auctions. By the revenue equivalence theorem, for rational contestants, expected total effort is invariant to θ and is identical under first- and second-price rules. We restate this result with our notation as follows.

Lemma 2 ***(Rational Benchmark: Total Effort)*** *Consider a contest with rational contestants ($\alpha = 0$). The following holds:*

(i) Both $TE^I(0, \theta)$ and $TE^{II}(0, \theta)$ are independent of $\theta \in [0, 1]$.

(ii) $TE^I(0, \theta) = TE^{II}(0, \theta)$ for all $\theta \in [0, 1]$.

Somewhat surprisingly, no existing study analyzes how the loser-pay fraction affects expected winner's effort or compares it across first- and second-price rules to our knowledge. We address this gap.

Lemma 3 ***(Rational Benchmark: Winner's Effort)*** *Consider a contest with rational contestants ($\alpha = 0$). The following holds:*

(i) Both $WE^I(0, \theta)$ and $WE^{II}(0, \theta)$ are strictly decreasing in $\theta \in [0, 1]$.

(ii) $WE^I(0, \theta) \geq WE^{II}(0, \theta)$ for all $\theta \in [0, 1]$, with equality holding if and only if $\theta = 0$.

By Lemma 3(i), with rational contestants, the loser-pay fraction θ reduces the expected winner's effort across both auction formats. Intuitively, a higher loser-pay fraction inflates the cost of bidding, thereby discouraging contestants from bidding aggressively and resulting in a lower expected winning bid.

Lemma 3(ii) indicates that discouragement due to the loser-pay feature is more pronounced under the second-price rule. To better understand the logic, note that equilibrium bids are initially higher in second-price contests for small θ. In the extreme case without the loser-pay feature—i.e., when $\theta = 0$—second-price bids equal valuations (no shading), whereas first-price bids are strictly shaded. Thus, the discouragement effect triggered by the proportional loser-pay component is more significant in second-price contests, which result in a larger bid reduction.

3 Expected Total Effort

In this section, we analyze the equilibrium expected total effort. Specifically, we examine how projection bias shapes its comparative statics with respect to the loser-pay fraction θ, and we compare this metric across the first- and second-price rules.

Proposition 1 ***(Comparative Statics of Total Effort)*** *For any fixed degree of projection bias $\alpha \in (0,1)$, the following comparative statics hold with respect to the loser-pay fraction $\theta \in [0,1]$:*

(i) Under the first-price rule, total effort $TE^I(\alpha, \theta)$ is independent of θ.

(ii) Under the second-price rule, total effort $TE^{II}(\alpha, \theta)$ is strictly increasing in θ.

Proposition 1 highlights how the interaction of projection bias and the winner-pay rule affects the total effort comparative statics with respect to the loser-pay fraction θ. In particular, projection bias and the second-price rule *in combination* cause expected total effort to increase in θ. To understand the underlying intuition, let us begin with the first-price rule. As θ increases, contestants naturally reduce their bids. Following standard payoff-equivalence logic, in the perceived game of a type-v contestant, interim expected effort remains invariant to θ.[16] Put differently, a type-v contestant adjusts his bid downward just enough to keep his perceived interim expected effort unchanged. Under the first-price rule, this perceived interim expected effort coincides exactly with the actual interim expected effort for type v, given by (7). This explains why the expected total effort does not change with θ.

Under the second-price rule, the payoff-equivalence logic still applies to the perceived game, so the type-v perceived interim expected effort does not vary with θ. However, due to projection bias, the perceived interim expected effort is $\int_{\underline{v}}^{v} \widetilde{\beta}^{II}\Big(\alpha v + (1-\alpha)t \mid v\Big) dH(t) + [1-H(v)]\,\theta\widehat{\beta}^{II}(v)$, where $\widetilde{\beta}^{II}\Big(\alpha v + (1-\alpha)t \mid v\Big)$ is the bid that the type-v contestant attributes to an opponent whose underlying valuation is $t \leq v$. This diverges from the actual interim expected effort in (8), where $\widehat{\beta}^{II}(t) = \widetilde{\beta}^{II}(t|t)$, and the divergence explains why the actual expected total effort becomes sensitive to θ.

To understand the monotonicity, we focus on the perception gap $\widetilde{\beta}^{II}\Big(\alpha v + (1-\alpha)t \mid v\Big) - \widetilde{\beta}^{II}(t|t)$ with $t \leq v$. Intuitively, this gap is positive, because a type-v contestant not only thinks a type-t opponent becomes a higher type $\alpha v + (1-\alpha)t$, but also thinks this opponent views the competition as being more intense, with v rather than t being the anchor for value distribution. As θ increases, a higher bid becomes more costly in the event of a loss. Lower-valued contestants therefore bid more cautiously, which makes their bids less sensitive to whether t or v anchors the perceived value distribution. Consequently, the

[16]In fact, applying the payoff-equivalence formula to the perceived game of a type-v contestant shows that the perceived interim payoff of type $t \in \mathcal{V}(v)$ is $U^I(t|v) = \int_{\underline{v}(v)}^{t} H\left(\frac{s-\alpha v}{1-\alpha}\right) ds$. Therefore, the perceived interim expected effort of type t, denoted by $\widetilde{e}^I(t|v)$, is $tH\left(\frac{t-\alpha v}{1-\alpha}\right) - U^I(t|v) = \int_{\underline{v}(v)}^{t} s\, dH\left(\frac{s-\alpha v}{1-\alpha}\right)$, where the equality follows from integration by parts. This expression does not depend on θ. It follows that the perceived interim expected effort for type v, $\widetilde{e}^I(v|v)$, is independent of θ.

perception gap narrows, which implies that the actual interim expected effort—and thus expected total effort—increases.

Next, we compare total effort across the two winner-pay rules. From Proposition 3 and Corollary 1 in GPR, we know that in the polar case of $\theta = 0$, revenue equivalence breaks down with projection-biased bidders and the first-price auction generates more revenue than its second-price counterpart. The comparison in the auction setting (i.e., $\theta = 0$) is straightforward: Projection bias leads to more aggressive bidding in a first-price auction, while it has no impact on bidders' equilibrium bidding strategy in a second-price auction.

However, the comparison incurs complications when losers also need to pay—i.e., when $\theta > 0$—because projection bias now alters contestants' equilibrium strategies under both formats (first- and second-price). Nonetheless, we show that the GPR ranking extends to any fixed $\theta \in [0, 1]$.

Proposition 2 ***(Total Effort Comparison)*** *For all $\alpha \in (0, 1)$ and $\theta \in [0, 1]$, we have that $TE^I(\alpha, \theta) > TE^{II}(\alpha, \theta)$.*

The intuition for this ranking is closely related to that of the comparative statics result. By payoff equivalence, the perceived interim expected effort remains identical under both the first- and second-price rules and is invariant to θ.[17] Because this perceived interim expected effort exceeds the actual interim expected effort under the second-price rule, the ranking result follows immediately.

4 Expected Winner's Effort

In this section, we analyze the equilibrium expected winner's effort, which corresponds to the quality of the completed project in the procurement setup.

Proposition 3 ***(Comparative Statics of Winner's Effort)*** *The following comparative-static results hold:*

(i) Fix any degree of projection bias $\alpha \in [0, 1)$. Under the first-price rule, the expected winner's effort $WE^I(\alpha, \theta)$ is decreasing in θ.

[17]For every $\theta \in [0, 1]$, by applying the payoff equivalence logic to the perceived game of a type-v contestant, we can conclude that the perceived interim expected effort satisfies $\widetilde{e}^I(v|v) = \widetilde{e}^{II}(v|v) = \int_{\underline{v}(v)}^{v} s \, dH\left(\frac{s-\alpha v}{1-\alpha}\right)$, which is independent of θ.

(ii) Under the second-price rule, the monotonicity of $WE^{II}(\alpha,\theta)$ with respect to θ is ambiguous. In particular, as $\alpha \to 1$, we have that[18]

$$\lim_{\theta\to 0^+} \frac{\partial WE^{II}(1,\theta)}{\partial\theta} = \mathbb{E}_{(2)}(V) - n\underline{v}, \text{ where } \mathbb{E}_{(2)}(V) = \int_{\underline{v}}^{\bar{v}} v dF_{(2)}(v), \text{ and} \tag{11}$$

$$\lim_{\theta\to 1^-} \frac{\partial WE^{II}(1,\theta)}{\partial\theta} = n\int_{\underline{v}}^{\bar{v}} v\,[1-F(v)]\log\Big(1-H(v)\Big)\left[1+\frac{\log(1-H(v))}{2}\right]dH(v). \tag{12}$$

The two partial derivatives can be positive or negative, depending on the value distribution $F(\cdot)$ and the number of contestants n.

Proposition 3(i) demonstrates that under the first-price rule, projection bias does not alter the monotonicity of the expected winner's effort in the loser-pay fraction θ: An increase in θ still sufficiently discourages bidding such that the expected winner's effort decreases. Proposition 3(ii), however, reveals that this monotonicity can be overturned under the second-price rule when contestants exhibit projection bias. Instead, the direction of this monotonicity depends intricately on the value distribution F and the number of contestants n.

The intuition that drives this contrast is as follows. As θ increases, losing becomes more costly; this discourages bidding, especially for low-valuation types. Meanwhile, there is a countervailing force: Because losing is more costly, the incentive to win becomes stronger, which intuitively affects the high types more than the low types. This countervailing force is stronger under the second-price rule than under the first-price rule, because upon winning the cost to the winner is not determined by his own bid, but rather the highest losing bid. In the rational case, however, the countervailing force is not strong enough even under the second-price rule, so the expected winner's effort unambiguously decreases with θ. When projection bias is present, it attenuates the direct bidding-cost effect by causing low types to perceive competition as less intense, while reinforcing the win-securing effect by causing high types to perceive competition as more intense. This reinforcement is particularly consequential under the second-price rule, where the win-securing effect is already stronger. As a result, the interaction between projection bias and the second-price payment structure can offset or even outweigh the direct bidding-cost effect, and the expected winner's effort under the second-price rule need not decrease monotonically in θ.

[18]Because our analysis is restricted to $\alpha \in [0,1)$, the value at the boundary $\alpha = 1$ is defined via its left-sided limit: $WE^{II}(1,\theta) := \lim_{\alpha\to 1^-} WE^{II}(\alpha,\theta)$.

The following example illustrates the conditions under which projection bias reinforces the win-securing effect strongly enough to overturn the direct bidding-cost effect under the second-price rule.

Example 1 *Fix* $\alpha \in [0,1)$ *and let* $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$ *on* $[\underline{v}, \bar{v}]$, *where* $0 < \underline{v} < \bar{v}$. *Then*

$$\lim_{\theta \to 0^+} \frac{\partial WE^{II}(\alpha,\theta)}{\partial \theta} > 0 \Leftrightarrow \alpha > \frac{n}{n+2} + \frac{2(n+1)}{(n+2)\left(\frac{\bar{v}}{\underline{v}} - 1\right)}.$$

The threshold degree of projection bias is below one if and only if $\bar{v}/\underline{v} > n+2$. Moreover, this threshold is decreasing in $\bar{v}/\underline{v}$ and increasing in n. Thus, the condition is more likely to be satisfied when the value distribution is more dispersed or when the contest involves fewer contestants. In either case, baseline rational competition is relatively mild, and thereby provides greater scope for projection bias to intensify high types' perceived competition and amplify the win-securing effect.

Despite the complexity of how the winner's effort is affected by the loser-pay fraction θ, we obtain the following unambiguous ranking across the two payment rules.

Proposition 4 ***(Winner's Effort Comparison)*** *For all* $\alpha \in [0,1)$ *and* $\theta \in [0,1]$, $WE^{I}(\alpha,\theta) \geq WE^{II}(\alpha,\theta)$, *with equality if and only if* $\alpha = \theta = 0$.

Proposition 4 establishes that the rational-benchmark ranking in Lemma 3 continues to hold under projection bias, which again indicates that the first-price rule is less affected by the proportional loser-pay component than the second-price rule due to the stronger bid-shading incentive.

5 Numerical Exercises

We complement our qualitative results with deterministic-quadrature exercises for $n = 2$, $\theta \in [0,1]$, and $\alpha \in \{0.1, 0.5, 0.9\}$. Figure 1 examines three value distributions on the support $[0,8]$: a uniform distribution; a truncated normal distribution $N(4,1)$ restricted to $[0,8]$; and a truncated lognormal distribution with $\log V \sim N(1.921, 1)$ bounded above at 8. All three distributions share a common mean of 4, with standard deviations of 2.31, 1.00, and 2.01, respectively. These comparisons thus hold both the support and the mean fixed while varying distributional shape and dispersion.

We first consider the comparative statics in θ under the second-price rule. Projection bias amplifies both the increase in total effort and the possible increase in the winner's

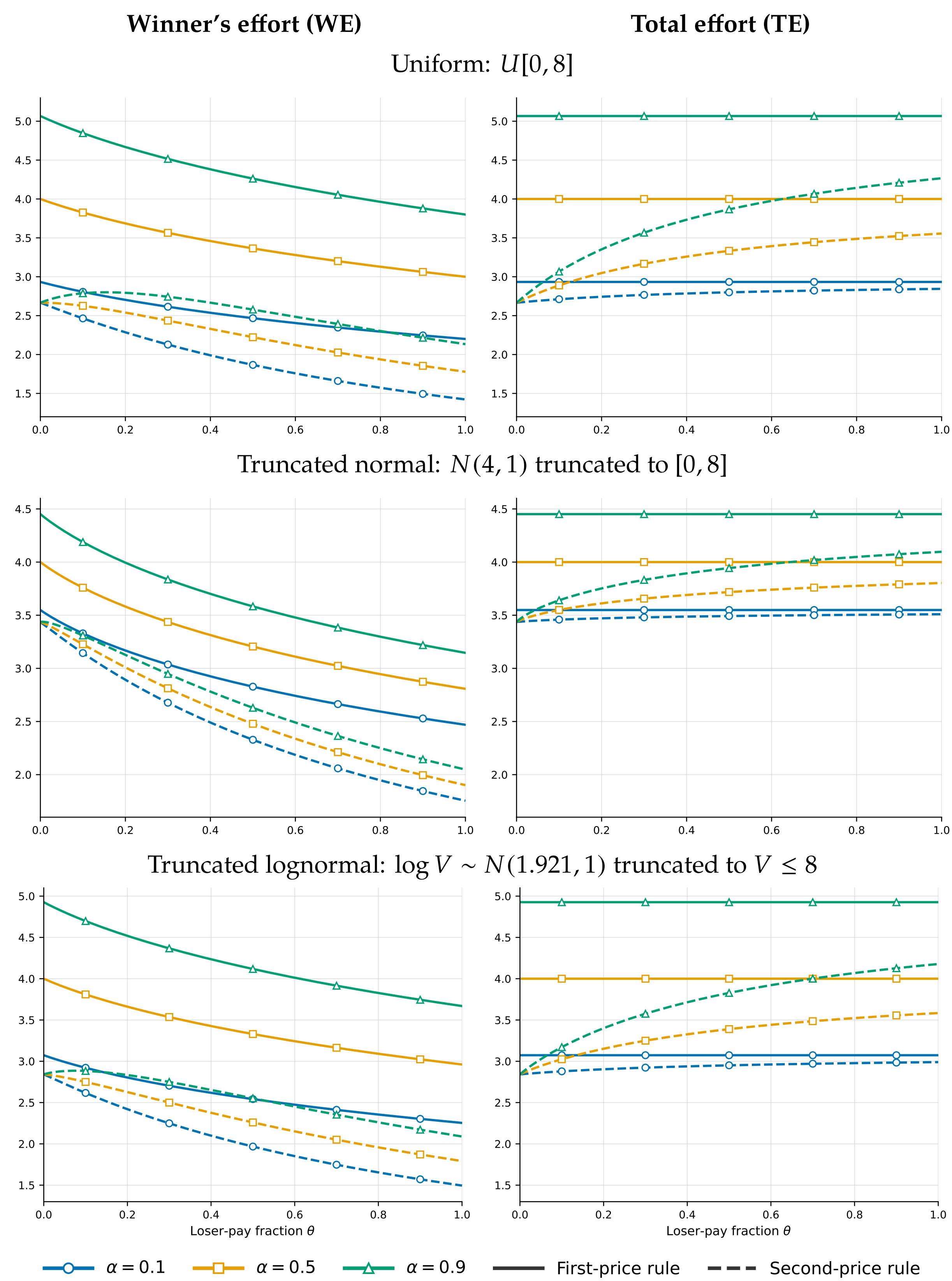

Figure 1: Expected winner's effort (left) and total effort (right) for two contestants.

effort. Raising θ from 0 to 1 leaves TE^{II} unchanged when $\alpha = 0$; across the uniform, truncated-normal, and truncated-lognormal distributions, it raises TE^{II} by 6.7%, 2.1%, and 5.2%, respectively, when $\alpha = 0.1$, but by 60.0%, 19.2%, and 47.0% when $\alpha = 0.9$. At $\alpha = 0.9$, the maximal gain in WE^{II} relative to $\theta = 0$ is 5.0% for the uniform distribution (at $\theta = 0.155$), less than 0.1% for the truncated normal, and 1.6% for the truncated lognormal (at $\theta = 0.076$). These quantitative effects are most pronounced for the two more dispersed distributions. Appendix B reports the corresponding figures for the uniform distribution with more contestants and shows that the effects attenuate as the number of bidders increases.[19]

Regarding cross-format comparisons, the advantage of the first-price rule in both the winner's effort and total effort widens with projection bias. At $\theta = 0$, the two effort gaps coincide: As α rises from 0.1 to 0.9, the gap—measured relative to the first-price outcome—increases from 3–9% to 23–47% across the three distributions. At $\theta = 1$, the winner-effort gap rises from 29–35% to 35–44%, while the total-effort gap expands from 1–3% to 8–16%.

6 Concluding Remarks

Using a contest framework, we study a procurement setting in which all suppliers make unconditional investments and examine how interpersonal projection shapes procurement outcomes. We conclude by discussing the implications of our results for procurement design.

First, in procurement, the buyer sometimes exercises discretion over the extent of preliminary work required. At one extreme, the buyer may demand fully developed, ready-to-use proposals or products; at the other, mere blueprints may suffice. Our comparative statics results with respect to the loser-pay fraction θ suggest that if the buyer seeks to maximize total investment, it is optimal to mandate complete upfront work before contract award. Conversely, if the buyer cares exclusively about the winner's investment or the quality of the completed project, then the optimal preliminary-work requirement depends jointly on the degree of projection bias, the competitiveness of the market, and the procurement format. In particular, raising the preliminary-work requirement can be desirable when market competition is limited, suppliers exhibit projection bias, and their progress during bid preparation is transparent.[20]

[19]For the uniform distribution at $\alpha = 0.9$, the TE^{II} increase from $\theta = 0$ to 1 falls from 60.0% for $n = 2$ to 33.4% for $n = 3$ and 23.2% for $n = 4$, while the maximal WE^{II} gain falls from 5.0% to 0.4% and less than 0.1%; Figure B1 reports the corresponding curves.

[20]As explained in the model setup, the first-price rule corresponds to a sealed-bid protocol in which the

Second, the buyer may also control the information flow during the bid preparation stage. We reaffirm the first-price advantage—established by GPR for standard auctions (i.e., $\theta = 0$) regarding total revenue—across any $\theta \in [0, 1]$ for both expected total effort and the expected winner's effort. These robustness results imply that the buyer should adopt a sealed-bid protocol to strictly prevent any information leakage concerning interim progress.

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suppliers cannot observe competitors' progress, whereas the second-price rule captures an open-progress protocol.

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Appendix A Proofs

We first state a useful intermediate result.

Lemma A1 *Let $g, h : [a, b] \to \mathbb{R}$ be continuous functions defined on the interval $[a, b]$. Suppose that*

(i) g is weakly increasing on $[a, b]$.

(ii) There exists a point $c \in (a, b)$ such that $h(x) \le 0$ for $x \in (a, c)$ and $h(x) \ge 0$ for $x \in (c, b)$.

Then the following inequality holds:

$$\int_a^b g(x)h(x)dx \ge g(c)\int_a^b h(x)dx.$$

Proof. When $a < x < c$, we have $g(x) \le g(c)$ and $h(x) \le 0$. When $c < x < b$, we have $g(x) \ge g(c)$ and $h(x) \ge 0$. In both cases, the product $\left[g(x) - g(c)\right] h(x) \ge 0$. Therefore, the integral of this non-negative product is non-negative:

$$\int_a^b g(x)h(x)dx - g(c)\int_a^b h(x)dx = \int_a^b \left[g(x) - g(c)\right] h(x)dx \ge 0,$$

which concludes the proof. ■

Proof of Lemma 1

Proof. Fix $\theta \in (0, 1)$. We first characterize the BNE for the rational case under both the first-price rule and the second-price rule. Consider the first-price contest. Suppose all other contestants use the BNE strategy $\beta^I(\cdot)$. Then the problem of a representative contestant with value v is

$$\max_t H(t)\left[v - \beta^I(t)\right] - [1 - H(t)]\,\theta\beta^I(t).$$

In equilibrium, the optimum must be achieved at $t = v$. Combining this with the first-order condition, we have that

$$H'(v)\left[v - (1-\theta)\beta^I(v)\right] - [(1-\theta)H(v) + \theta]\frac{d}{dv}\beta^I(v) = 0.$$

It is straightforward to verify that (4) is the solution to the differential equation with boundary condition $\beta^I(\underline{v}) = 0$.

Consider the second-price auction. The problem of a representative contestant with value v is

$$\max_t \int_{\underline{v}}^{t} \left[v - \beta^{II}(s)\right] dH(s) - [1 - H(t)]\, \theta \beta^{II}(t).$$

Substituting $t = v$ into the first-order condition yields

$$H'(v) \left[v - (1-\theta)\beta^{II}(v)\right] - \theta \left[1 - H(v)\right] \frac{d}{dv}\beta^{II}(v) = 0.$$

The solution of the resulting differential equation with boundary condition $\beta^{II}(\underline{v}) = 0$ is exactly (5).

Now we proceed to the case with projection bias and derive the NBE strategy for the first-price contest and the second-price contest. Consider contestant i with valuation $v_i \in \mathcal{V}$. He perceives that the value distribution is $\widehat{F}(\cdot|v_i)$ given by (1). The NBE of a contest can be obtained from the BNE bidding strategy obtained for the rational case by replacing $F(\cdot)$ with the perceived distribution function $\widehat{F}(\cdot|v_i)$, then evaluating the new bidding function at v_i.

Specifically, for the first-price contest, we have that

$$\widehat{\beta}^{I}(v_i) = \frac{\int_{\underline{v}(v_i)}^{v_i} t dH\left(\frac{t-\alpha v_i}{1-\alpha}\right)}{(1-\theta)H(v_i)+\theta} \overset{s=\frac{t-\alpha v_i}{1-\alpha}}{=\!=\!=} \frac{\int_{\underline{v}}^{v_i} \left(\alpha v_i + (1-\alpha)s\right) dH(s)}{(1-\theta)H(v_i)+\theta} = \alpha G^{I}(v_i) + (1-\alpha)\beta^{I}(v_i).$$

For the second-price contest, we have that

$$\begin{aligned}
\widehat{\beta}^{II}(v_i) &= \frac{[1-H(v_i)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}(v_i)}^{v_i} \frac{t}{\left[1 - H\left(\frac{t-\alpha v_i}{1-\alpha}\right)\right]^{\frac{1}{\theta}}} dH\left(\frac{t-\alpha v_i}{1-\alpha}\right) \\
&\overset{s=\frac{t-\alpha v_i}{1-\alpha}}{=\!=\!=} \frac{[1-H(v_i)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}}^{v_i} \frac{(\alpha v_i + (1-\alpha)s)}{[1-H(s)]^{\frac{1}{\theta}}} dH(s) \\
&= \alpha \left[\frac{[1-H(v_i)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}}^{v_i} \frac{v_i}{[1-H(s)]^{\frac{1}{\theta}}} dH(s)\right] + (1-\alpha)\beta^{II}(v_i) \\
&= \alpha G^{II}(v_i) + (1-\alpha)\beta^{II}(v_i).
\end{aligned}$$

It remains to consider the boundary cases. For $\theta = 0$, GPR directly implies that

$$\widehat{\beta}^{I}(v_i) = \alpha v_i + (1-\alpha)\frac{\int_{\underline{v}}^{v_i} t\, dH(t)}{H(v_i)} \qquad \text{and} \qquad \widehat{\beta}^{II}(v_i) = v_i.$$

When $\theta = 1$, the two first-order conditions reduce to

$$(\beta^I)'(v) = vH'(v) \qquad \text{and} \qquad \Big[1 - H(v)\Big](\beta^{II})'(v) = vH'(v).$$

Using the boundary condition whereby the lowest type bids zero yields

$$\beta^I(v) = \int_{\underline{v}}^{v} t\, dH(t) \qquad \text{and} \qquad \beta^{II}(v) = \int_{\underline{v}}^{v} \frac{t}{1 - H(t)}\, dH(t).$$

Applying the same change of distribution as above gives

$$\widehat{\beta}^I(v_i) = \alpha v_i H(v_i) + (1 - \alpha) \int_{\underline{v}}^{v_i} t\, dH(t)$$

and

$$\widehat{\beta}^{II}(v_i) = -\alpha v_i \log\Big(1 - H(v_i)\Big) + (1 - \alpha) \int_{\underline{v}}^{v_i} \frac{t}{1 - H(t)}\, dH(t).$$

Finally, simple algebra would verify that these boundary equilibria coincide with the corresponding one-sided limits of the equilibrium strategies derived above for $\theta \in (0, 1)$. Therefore, the expressions in Equations (2) to (5) continue to apply for $\theta \in [0, 1]$ when their values at $\theta = 0$ and $\theta = 1$ are defined by these continuous extensions.[1] This concludes the proof. ■

Proof of Corollary 1

Proof. It suffices to show that $G^k(v) > \beta^k(v)$ for $k \in \{I, II\}$ and $v \in (\underline{v}, \bar{v})$. In fact, we have that

$$G^I(v) - \beta^I(v) = \frac{\int_{\underline{v}}^{v} (v - s) dH(s)}{(1 - \theta)H(v) + \theta} > 0,$$

for all $\theta \in [0, 1]$ and

$$G^{II}(v) - \beta^{II}(v) = \frac{[1 - H(v)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}}^{v} \frac{v - s}{[1 - H(s)]^{\frac{1}{\theta}}} dH(s) > 0$$

for all $\theta \in (0, 1]$. This concludes the proof. ■

[1] The only qualification arises at $v = \underline{v}$ when $\underline{v} > 0$: The limits as $v \to \underline{v}^+$ and $\theta \to 0^+$ do not commute because the equilibrium for every $\theta > 0$ assigns a zero bid to the lowest type, whereas at $\theta = 0$ that type is indifferent among all bids in $[0, \underline{v}]$; this is merely an equilibrium-selection issue at a zero-probability type and does not affect any expected outcome.

Proof of Lemma 2

Proof. The lemma follows immediately from the revenue equivalence theorem. ■

Proof of Lemma 3

Proof. We first prove Part (i) of the lemma. For the first-price rule, by the definition of the winner's effort, we have

$$WE^I(0,\theta) = \int_{\underline{v}}^{\bar{v}} \beta^I(v) dF^n(v) = \int_{\underline{v}}^{\bar{v}} dF^n(v) \int_{\underline{v}}^{v} \frac{t}{(1-\theta)H(v)+\theta} dH(t). \tag{13}$$

The monotonicity is obvious by (13).

For the second-price rule, recall that $F_{(2)}(v) = F^n(v) + nF^{n-1}(v)(1-F(v))$ denotes the cumulative distribution function of the second-highest valuation among n independent and identically distributed random variables with distribution F. By the definition of expected winner's effort and $dF_{(2)}(v) = n(n-1)F^{n-2}(v)(1-F(v))dF(v)$, we have

$$\begin{aligned}
WE^{II}(0,\theta) &= \int_{\underline{v}}^{\bar{v}} \beta^{II}(v) dF_{(2)}(v) \\
&= \int_{\underline{v}}^{\bar{v}} \left[\frac{[1-H(v)]^{\frac{1-\theta}{\theta}}}{\theta} \int_{\underline{v}}^{v} \frac{t dH(t)}{[1-H(t)]^{\frac{1}{\theta}}} \right] n(n-1)F^{n-2}(v)(1-F(v))dF(v) \\
&= \int_{\underline{v}}^{\bar{v}} \frac{[1-H(v)]^{\frac{1-\theta}{\theta}}}{\theta} \left[\int_{\underline{v}}^{v} \frac{t}{[1-H(t)]^{\frac{1}{\theta}}} \int_{v}^{\bar{v}} dF(u) dH(t) \right] n dH(v) \\
&= \int_{\underline{v}}^{\bar{v}} n dF(u) \int_{\underline{v}}^{u} \frac{t dH(t)}{[1-H(t)]^{\frac{1}{\theta}}} \int_{t}^{u} \frac{[1-H(v)]^{\frac{1-\theta}{\theta}}}{\theta} dH(v) \\
&= \int_{\underline{v}}^{\bar{v}} \frac{dF^n(u)}{H(u)} \int_{\underline{v}}^{u} \frac{t}{[1-H(t)]^{\frac{1}{\theta}}} \left[(1-H(t))^{\frac{1}{\theta}} - (1-H(u))^{\frac{1}{\theta}} \right] dH(t) \\
&= \int_{\underline{v}}^{\bar{v}} dF^n(u) \int_{\underline{v}}^{u} \frac{t}{H(u)} \left[1 - \left(\frac{1-H(u)}{1-H(t)} \right)^{\frac{1}{\theta}} \right] dH(t).
\end{aligned} \tag{14}$$

Therefore, the monotonicity follows from the fact that $\left(\frac{1-H(u)}{1-H(t)}\right)^{\frac{1}{\theta}}$ is increasing in θ for $t < u$.

Next, we prove Part (ii). From Equations (13) and (14), we have

$$WE^I(0,\theta)-WE^{II}(0,\theta) = \int_{\underline{v}}^{\bar{v}} dF^n(u) \int_{\underline{v}}^{u} \frac{t}{H(u)} \left[\left(\frac{1-H(u)}{1-H(t)} \right)^{\frac{1}{\theta}} - \frac{\theta(1-H(u))}{(1-\theta)H(u)+\theta} \right] dH(t).$$

It suffices to show that for $u \in (\underline{v}, \bar{v})$, the following inequality holds, with equality if and only if $\theta = 0$:

$$\int_{\underline{v}}^{u} \frac{t}{H(u)} \left[\left(\frac{1-H(u)}{1-H(t)} \right)^{\frac{1}{\theta}} - \frac{\theta(1-H(u))}{(1-\theta)H(u)+\theta} \right] dH(t) \geq 0.$$

It is straightforward to verify that equality holds when $\theta = 0$, which is consistent with revenue equivalence (since the winner's effort is essentially revenue in that case), and strict inequality holds when $\theta = 1$. Now consider the case when $0 < \theta < 1$.

Fix $u \in (\underline{v}, \bar{v})$. Let $g(t) = \frac{t}{H(u)}$ and $h(t) = \left(\frac{1-H(u)}{1-H(t)}\right)^{\frac{1}{\theta}} - \frac{\theta(1-H(u))}{(1-\theta)H(u)+\theta}$. We need to show that $\int_{\underline{v}}^{u} g(t)h(t)H'(t)dt > 0$. It is obvious that $h(t)$ is increasing in t. Moreover, we have that

$$h(\underline{v}) = (1-H(u)) \left[(1-H(u))^{\frac{1-\theta}{\theta}} - \frac{1}{\frac{1-\theta}{\theta}H(u)+1} \right] \leq 0,$$

where the inequality follows from Bernoulli's inequality, and it is straightforward to see that $h(u) \geq 0$. Therefore, there exists t_0 such that $h(t)H'(t) \leq 0$ for $t \in (\underline{v}, t_0)$ and $h(t)H'(t) \geq 0$ for $t \in (t_0, u)$. By Lemma A1, we have that

$$\begin{aligned}
\int_{\underline{v}}^{u} g(t)h(t)H'(t)dt &\geq g(t_0) \int_{\underline{v}}^{u} h(t)H'(t)dt \\
&= g(t_0) \int_{\underline{v}}^{u} \left[\left(\frac{1-H(u)}{1-H(t)} \right)^{\frac{1}{\theta}} - \frac{\theta(1-H(u))}{(1-\theta)H(u)+\theta} \right] dH(t) \\
&= g(t_0) \left\{ \left[\frac{\theta(1-H(u))^{\frac{1}{\theta}}}{(1-\theta)(1-H(t))^{\frac{1-\theta}{\theta}}} \right] \Bigg|_{\underline{v}}^{u} - \frac{\theta H(u)(1-H(u))}{(1-\theta)H(u)+\theta} \right\} \\
&= \frac{\theta g(t_0)(1-H(u))}{1-\theta} \left[\frac{1}{\frac{1-\theta}{\theta}H(u)+1} - (1-H(u))^{\frac{1-\theta}{\theta}} \right] > 0,
\end{aligned}$$

where the last inequality follows from Bernoulli's inequality. ■

Proof of Proposition 1

Proof. By Equations (2) and (3), a contestant's expected effort can be decomposed into two parts:

$$e^I(v) = \alpha e_p^I(v) + (1-\alpha)e_r^I(v), \quad e^{II}(v) = \alpha e_p^{II}(v) + (1-\alpha)e_r^{II}(v),$$

where

$$e_p^I(v) = H(v)G^I(v) + (1-H(v))\theta G^I(v), \quad e_r^I(v) = H(v)\beta^I(v) + (1-H(v))\theta\beta^I(v),$$
$$e_p^{II}(v) = \int_{\underline{v}}^{v} G^{II}(t)dH(t) + (1-H(v))\theta G^{II}(v), \quad e_r^{II}(v) = \int_{\underline{v}}^{v} \beta^{II}(t)dH(t) + (1-H(v))\theta\beta^{II}(v).$$

For $k \in \{I, II\}$, $e_r^k(v)$ represents the expected effort in the k-th price contest when the contestant with valuation v is rational ($\alpha = 0$), and $e_p^k(v)$ is the expected effort when the contestant is extremely projection-biased ($\alpha \to 1^-$). For $\alpha \in (0,1)$, the expected effort is a weighted average of these two special cases. By revenue equivalence (Lemma 2), we know that $e_r^I(v) = e_r^{II}(v) = \int_{\underline{v}}^{v} t dH(t)$ for all $\theta \in [0,1]$. Therefore, we only need to consider $e_p^k(v)$. Specifically, it is sufficient to prove that for all $v \in \mathcal{V}$, $e_p^I(v)$ is independent of θ, and $e_p^{II}(v)$ is increasing in θ.

We first prove Part (i) of the proposition. In fact, $e_p^I(v)$ is independent of θ because

$$e_p^I(v) = H(v)G^I(v) + (1-H(v))\theta G^I(v) = [(1-\theta)H(v) + \theta]\, G^I(v) = vH(v).$$

Next, we prove Part (ii). We proceed to show that $e_p^{II}(v)$ is increasing in θ. In fact, we have that

$$\begin{aligned}
e_p^{II}(v) &= \int_{\underline{v}}^{v} G^{II}(t)dH(t) + (1-H(v))\theta G^{II}(v) \\
&= \int_{\underline{v}}^{v} \left[G^{II}(t)H'(t) + \theta\left(G^{II}\right)'(t) - \theta H(t)\left(G^{II}\right)'(t) - \theta H'(t)G^{II}(t)\right] dt \\
&= \int_{\underline{v}}^{v} \left[(1-\theta)G^{II}(t)H'(t) + \theta\left(G^{II}\right)'(t)(1-H(t))\right] dt \\
&= \int_{\underline{v}}^{v} (1-\theta)G^{II}(t)H'(t)dt \\
&\quad + \int_{\underline{v}}^{v} \theta(1-H(t))\left[\frac{t(1-H(t))^{\frac{1}{\theta}-2}H'(t)}{\theta} + \frac{1-(1-H(t))^{\frac{1-\theta}{\theta}}}{1-\theta}\right] dt
\end{aligned}$$

$$= \int_{\underline{v}}^{v} t dH(t) + \int_{\underline{v}}^{v} \theta(1-H(t)) \frac{1-(1-H(t))^{\frac{1-\theta}{\theta}}}{1-\theta} dt.$$

Let $u = \frac{1-\theta}{\theta}$ and $a = 1 - H(t)$. It is sufficient to prove that for $a \in (0,1)$ and $u > 0$, $\delta(u) = \frac{1-a^u}{u}$ is decreasing in u. Taking the derivative, we have that $\delta'(u) = -\frac{1-a^u+ua^u \log a}{u^2}$. Let $\eta(u) = 1 - a^u + ua^u \log a$. We have that $\eta'(u) = ua^u \left(\log a\right)^2 > 0$. Therefore, $\eta(u) > \eta(0) = 0$ for $u \in (0, +\infty)$, which implies that $\delta'(u) < 0$ for $u \in (0, +\infty)$. ■

Proof of Proposition 2

Proof. From Part (ii) of the proof of Proposition 1, we know that $e_p^{II}(v)$ is increasing in θ. By the decomposition of $e^k(v)$ with $k \in \{I, II\}$ in the proof of Proposition 1, it suffices to show that $e_p^I(v)$ (which is independent of θ) is greater than $e_p^{II}(v)$ when $\theta = 1$. In fact, by L'Hôpital's rule, we have

$$\lim_{\theta \to 1^-} e_p^{II}(v) = \int_{\underline{v}}^{v} t dH(t) + \int_{\underline{v}}^{v} -(1-H(t)) \log(1-H(t)) dt.$$

Integration by parts yields that

$$e_p^I(v) = vH(v) = \int_{\underline{v}}^{v} t dH(t) + \int_{\underline{v}}^{v} H(t) dt.$$

Let $u = H(t) \in (0,1)$. The fact that $u > -(1-u)\log(1-u)$ completes the proof. ■

Proof of Proposition 3

Proof. Analogous to the proof of Proposition 1, the winner's effort can be decomposed into two parts:

$$\begin{aligned} WE^I(\alpha,\theta) &= \alpha WE_p^I(\theta) + (1-\alpha) WE_r^I(\theta), \\ WE^{II}(\alpha,\theta) &= \alpha WE_p^{II}(\theta) + (1-\alpha) WE_r^{II}(\theta), \end{aligned} \tag{15}$$

where

$$WE_p^I(\theta) := \lim_{\alpha \to 1^-} WE^I(\alpha,\theta) = \int_{\underline{v}}^{\bar{v}} G^I(v)\, dF^n(v), \quad WE_r^I(\theta) := WE^I(0,\theta) = \int_{\underline{v}}^{\bar{v}} \beta^I(v) dF^n(v),$$

$$WE_p^{II}(\theta) := \lim_{\alpha \to 1^-} WE^{II}(\alpha,\theta) = \int_{\underline{v}}^{\bar{v}} G^{II}(v)\, dF_{(2)}(v), \quad WE_r^{II}(\theta) := WE^{II}(0,\theta) = \int_{\underline{v}}^{\bar{v}} \beta^{II}(v) dF_{(2)}(v).$$

We first prove Part (i) of the proposition. From Lemma 3(i), we know that $WE_r^I(\theta)$ is decreasing in θ. By the decomposition, it is sufficient to show that $WE_p^I(\theta)$ is decreasing in θ. This can be seen clearly from the following equation:

$$WE_p^I(\theta) = \int_{\underline{v}}^{\bar{v}} G^I(v)dF^n(v) = \int_{\underline{v}}^{\bar{v}} \frac{vH(v)}{(1-\theta)H(v)+\theta}dF^n(v). \tag{16}$$

Next, we prove Part (ii). It suffices to prove the second sentence of this part, since (11) and (12) imply that the sign of $\lim\limits_{\theta\to 0^+} \frac{\partial WE^{II}(1,\theta)}{\partial\theta}$ and $\lim\limits_{\theta\to 1^-} \frac{\partial WE^{II}(1,\theta)}{\partial\theta}$ can be either positive or negative. We have that

$$\begin{aligned} WE_p^{II}(\theta) &= \int_{\underline{v}}^{\bar{v}} \frac{v}{1-\theta}\left[1-(1-H(v))^{\frac{1-\theta}{\theta}}\right] n(n-1)F^{n-2}(v)(1-F(v))dF(v) \\ &= n\int_0^1 \left(1-y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) \frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{1-\theta}dy, \end{aligned}$$

where the second equality follows from the change of variable $y = H(v) = F^{n-1}(v)$ and $dy = (n-1)F^{n-2}(v)dF(v)$. Let

$$I(\theta) := \frac{1}{n}WE_p^{II}(\theta) = \int_0^1 \left(1-y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) \phi(y,\theta)dy,$$

where $\phi(y,\theta) := \frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{1-\theta}$ and $P(x) := F^{-1}(x)$. We aim to solve for $\lim_{\theta\to 0^+} I'(\theta)$ and $\lim_{\theta\to 1^-} I'(\theta)$. The statement follows from the fact that $WE_p^{II}(\theta) = nI(\theta)$.

Calculation of $\lim_{\theta\to 0^+} I'(\theta)$. We first show that for all $\theta \in (0,1)$,

$$I'(\theta) = \int_0^1 \left(1-y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) \frac{\partial\phi}{\partial\theta}dy. \tag{17}$$

In fact, for $\theta \in (0,1)$, $y \in (0,1)$,

$$\frac{\partial\phi}{\partial\theta} = \mu_1(y,\theta) + \mu_2(y,\theta) \tag{18}$$

where

$$\mu_1(y,\theta) = \frac{(1-y)^{\frac{1-\theta}{\theta}}\log(1-y)}{\theta^2(1-\theta)},\ \mu_2(y,\theta) = \frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{(1-\theta)^2}.$$

Given any $\theta_0 \in (0,1)$ and for some small ε, $\left|\frac{\partial\phi}{\partial\theta}\right|$ is uniformly bounded in $[\theta_0-\varepsilon, \theta_0+\varepsilon]$. Therefore, Equation (17) holds by the dominated convergence theorem (DCT).

Now we proceed to calculate $\lim_{\theta\to 0^+} I'(\theta)$. Define $Q(x) = P(x) - P(0)$. By (17),

$$\begin{aligned} I'(\theta) &= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right)\left[Q\left(y^{\frac{1}{n-1}}\right) + P(0)\right]\left(\mu_1(y,\theta) + \mu_2(y,\theta)\right) dy \\ &\equiv J_1(\theta) + P(0) J_2(\theta), \end{aligned} \tag{19}$$

where

$$J_1(\theta) := \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right)\left(\mu_1(y,\theta) + \mu_2(y,\theta)\right) dy$$
$$J_2(\theta) := \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right)\left(\mu_1(y,\theta) + \mu_2(y,\theta)\right) dy$$

Next, we derive the expressions of $\lim_{\theta\to 0^+} J_1(\theta)$ and $\lim_{\theta\to 0^+} J_2(\theta)$.

Step 1: Calculation of $\lim_{\theta\to 0^+} J_1(\theta)$. Before the calculation, we evaluate the limits of μ_1 and μ_2: For $y \in (0,1)$,

$$\lim_{\theta\to 0^+} \mu_1(y,\theta) = 0 \text{ and } \lim_{\theta\to 0^+} \mu_2(y,\theta) = 1.$$

We first show that for $J_1(\theta)$, the order of taking the limit and evaluating the integral can be exchanged. In fact, since $\mu_2(y,\theta)$ is uniformly bounded in y when $\theta \to 0^+$, by DCT we have

$$\begin{aligned} \lim_{\theta\to 0^+} \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) \mu_2(y,\theta) dy &= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) \lim_{\theta\to 0^+} \mu_2(y,\theta) dy \\ &= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) dy. \end{aligned} \tag{20}$$

For the μ_1 part, for $\theta \in \left(0, \frac{1}{2}\right)$ we have

$$\begin{aligned} \left|\mu_1(y,\theta)\right| &= \left|\frac{(1-y)^{\frac{1-\theta}{\theta}} \log(1-y)}{\theta^2(1-\theta)}\right| = \left|\frac{\log(1-y)}{(1-\theta)}\right| \frac{(1-y)^{\frac{1-\theta}{\theta}}}{\theta^2} \\ &\le \left|\frac{\log(1-y)}{(1-\theta)}\right| \frac{(1-y)^{\frac{1}{2\theta}}}{\theta^2} = \left|\frac{4}{\log(1-y)(1-\theta)}\right| \left[\frac{\left(\log(1-y)\right)}{2\theta}\right]^2 \exp\left[\frac{\log(1-y)}{2\theta}\right] \\ &\le 8 \left|\frac{1}{\log(1-y)}\right| \sup_{x\ge 0} x^2 e^{-x} = C \left|\frac{1}{\log(1-y)}\right| \end{aligned}$$

for some $C > 0$. To use DCT, we need to prove that $\left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) \left|\frac{1}{\log(1-y)}\right|$ is integrable in $(0, 1)$. We only need to consider the integrability near $y \to 0^+$. When $y \to 0^+$, $|\log(1 - y)| \sim y$. Because $Q'(0) = 1/f(\underline{v}) < \infty$, we have that $\left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right)/y$ is integrable in $(0, 1)$. Therefore, by DCT, we have

$$\lim_{\theta\to 0^+} \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) \mu_1(y, \theta) dy = \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) \lim_{\theta\to 0^+} \mu_1(y, \theta) dy = 0. \tag{21}$$

Combining Equations (20) and (21) yields that

$$\begin{aligned}
\lim_{\theta\to 0^+} J_1(\theta) &= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) Q\left(y^{\frac{1}{n-1}}\right) dy \\
&= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) \left[P\left(y^{\frac{1}{n-1}}\right) - P(0)\right] dy \\
&= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) dy - \frac{P(0)}{n} \\
&= \int_{\underline{v}}^{\bar{v}} v(n-1)F^{n-2}(v)(1 - F(v))dF(v) - \frac{\underline{v}}{n} = \frac{E_{(2)}(V) - \underline{v}}{n},
\end{aligned} \tag{22}$$

where the fourth equality follows from $y = H(v)$.

Step 2: Calculation of $\lim_{\theta\to 0^+} J_2(\theta)$. For the μ_2 part, the condition for DCT is still satisfied, and we have

$$\lim_{\theta\to 0^+} \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) \mu_2(y, \theta) dy = \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) \lim_{\theta\to 0^+} \mu_2(y, \theta) dy = \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) dy = \frac{1}{n}. \tag{23}$$

However, for the μ_1 part, we cannot use DCT to exchange the order of limit and integral. Now we calculate

$$\lim_{\theta\to 0^+} \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) \mu_1(y, \theta) dy = \lim_{\theta\to 0^+} \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) \frac{(1-y)^{\frac{1-\theta}{\theta}} \log(1-y)}{\theta^2(1-\theta)} dy.$$

Let

$$u = -\frac{\log(1-y)}{\theta}, \quad \text{so that} \quad y = 1 - e^{-\theta u}, \quad dy = \theta e^{-\theta u}\, du.$$

Thus,

$$\int_0^1 \frac{\left(1-y^{\frac{1}{n-1}}\right)(1-y)^{\frac{1-\theta}{\theta}}\log(1-y)}{\theta^2(1-\theta)}\,dy = -\frac{1}{(1-\theta)}\int_0^{+\infty} u\left[1-\left(1-e^{-\theta u}\right)^{\frac{1}{n-1}}\right]e^{-u}\,du.$$

Therefore, we have

$$\begin{aligned}\lim_{\theta\to 0^+}\int_0^1\left(1-y^{\frac{1}{n-1}}\right)\mu_1(y,\theta)dy &= \lim_{\theta\to 0^+} -\frac{1}{(1-\theta)}\int_0^{+\infty} u\left[1-\left(1-e^{-\theta u}\right)^{\frac{1}{n-1}}\right]e^{-u}\,du\\ &= -\int_0^{+\infty} ue^{-u}du = -1.\end{aligned} \tag{24}$$

Therefore, combining Equations (23) and (24), we have

$$\lim_{\theta\to 0^+} J_2(\theta) = \frac{1}{n} - 1. \tag{25}$$

By Equations (19), (22), and (25), we can obtain that

$$\lim_{\theta\to 0^+} I'(\theta) = \frac{E_{(2)}(V)-\underline{v}}{n} + \underline{v}\left(\frac{1}{n}-1\right) = \frac{E_{(2)}(V)-n\underline{v}}{n}.$$

Calculation of $\lim_{\theta\to 1^-} I'(\theta)$. We can exchange the order of taking the derivative and integral by Equation (17). For the order exchange of the limit and the integral, we need to find a control function $\nu(y)$ such that for all $\theta \in \left(\frac{1}{2},1\right)$ and $y\in(0,1)$,

$$\left|\frac{\partial\phi}{\partial\theta}\right| < \nu(y).$$

In fact, the following function satisfies the condition above:

$$\nu(y) = 4\left[\log^2(1-y) - \log(1-y)\right].$$

We proceed to prove this. Define

$$\lambda(y,\theta) := \theta^2(1-\theta)^2\frac{\partial\phi}{\partial\theta} = (1-\theta)(1-y)^{\frac{1-\theta}{\theta}}\log(1-y) + \theta^2\left[1-(1-y)^{\frac{1-\theta}{\theta}}\right].$$

From Equation (18), $\lambda(y, \theta)$ can be decomposed as follows:

$$\lambda(y, \theta) = \frac{1}{2}\left[\zeta_1(y, \theta) + \zeta_2(y, \theta)\right],$$

where

$$\zeta_1(y, \theta) = \lambda(y, \theta) - (1-\theta)^2\left[\log^2(1-y) - \log(1-y)\right],$$
$$\zeta_2(y, \theta) = \lambda(y, \theta) + (1-\theta)^2\left[\log^2(1-y) - \log(1-y)\right].$$

We have that $\zeta_1(0, \theta) = \zeta_2(0, \theta) = 0$. Moreover, we have that $\frac{\partial}{\partial y}\zeta_1(y, \theta) < 0$ and $\frac{\partial}{\partial y}\zeta_2(y, \theta) > 0$. In fact, taking the derivative, we have

$$\frac{\partial}{\partial y}\zeta_1(y, \theta) = -(1-\theta)^2(1-y)^{\frac{1}{\theta}-2}\left[1 + \frac{1}{\theta}\log(1-y)\right] - (1-\theta)^2\frac{\left[1 - 2\log(1-y)\right]}{(1-y)},$$

$$\frac{\partial}{\partial y}\zeta_2(y, \theta) = -(1-\theta)^2(1-y)^{\frac{1}{\theta}-2}\left[1 + \frac{1}{\theta}\log(1-y)\right] + (1-\theta)^2\frac{\left[1 - 2\log(1-y)\right]}{(1-y)}.$$

So $\frac{\partial}{\partial y}\zeta_1(y, \theta) < 0$ and $\frac{\partial}{\partial y}\zeta_2(y, \theta) > 0$ are equivalent to

$$\left|(1-y)^{\frac{1}{\theta}-1}\left[1 + \frac{1}{\theta}\log(1-y)\right]\right| < 1 - 2\log(1-y),$$

which holds because $0 < (1-y)^{\frac{1}{\theta}-1} < 1$ and $\left|1 + \frac{1}{\theta}\log(1-y)\right| \leq 1 - \frac{1}{\theta}\log(1-y) < 1 - 2\log(1-y)$ for $\theta \in \left(\frac{1}{2}, 1\right)$.

Therefore, we know that for all $y \in (0, 1)$ and $\theta \in \left(\frac{1}{2}, 1\right)$, $\zeta_1(y, \theta) < 0$ and $\zeta_2(y, \theta) > 0$, which implies that $\left|\lambda(y, \theta)\right| \leq (1-\theta)^2\left[\log^2(1-y) - \log(1-y)\right]$ and

$$\left|\frac{\partial \phi}{\partial \theta}\right| = \left|\frac{\lambda(y, \theta)}{\theta^2(1-\theta)^2}\right| \leq \frac{\log^2(1-y) - \log(1-y)}{\theta^2} \leq 4\left[\log^2(1-y) - \log(1-y)\right] = \nu(y).$$

Therefore, by DCT, we have that

$$\begin{aligned}
\lim_{\theta \to 1^-} I'(\theta) &= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) \lim_{\theta \to 1^-} \frac{\partial \phi}{\partial \theta} dy \\
&= \int_0^1 \left(1 - y^{\frac{1}{n-1}}\right) P\left(y^{\frac{1}{n-1}}\right) \left[\frac{1}{2}\log(1-y)\left(2 + \log(1-y)\right)\right] dy \\
&= \int_{\underline{v}}^{\bar{v}} v\left[1 - F(v)\right] \log(1 - H(v)) \left[1 + \frac{\log(1 - H(v))}{2}\right] dH(v).
\end{aligned}$$

This completes the whole proof. ■

Proof of Example 1

Proof. By (15),

$$\lim_{\theta\to 0^+}\frac{\partial WE^{II}(\alpha,\theta)}{\partial\theta} = \alpha\lim_{\theta\to 0^+}\frac{\partial WE_p^{II}(\theta)}{\partial\theta} + (1-\alpha)\lim_{\theta\to 0^+}\frac{\partial WE_r^{II}(\theta)}{\partial\theta}.$$

For the first term, (11) and the expectation of the second-highest value under the uniform distribution give

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{\partial WE_p^{II}(\theta)}{\partial\theta} &= \mathbb{E}_{(2)}(V) - n\underline{v}\\ &= \underline{v} + \frac{n-1}{n+1}(\bar{v}-\underline{v}) - n\underline{v}\\ &= \frac{n-1}{n+1}\Big[\bar{v} - (n+2)\underline{v}\Big]. \end{aligned}\tag{26}$$

We next calculate the second term. From (14) and $dF^n(u) = nH(u)dF(u)$, differentiation for $\theta \in (0,1)$ yields

$$\frac{\partial WE_r^{II}(\theta)}{\partial\theta} = n\int_{\underline{v}}^{\bar{v}} dF(u)\int_{\underline{v}}^{u} t\left(\frac{1-H(u)}{1-H(t)}\right)^{\frac{1}{\theta}}\frac{\log\left(\frac{1-H(u)}{1-H(t)}\right)}{\theta^2}dH(t).$$

The differentiation is valid on every compact subinterval of $(0,1)$ because the value support is bounded and the integrand and its derivative are dominated there. In the inner integral, use

$$z = -\frac{1}{\theta}\log\left(\frac{1-H(u)}{1-H(t)}\right).$$

Because

$$1-H(t) = [1-H(u)]e^{\theta z},$$

we obtain

$$\begin{aligned}\frac{\partial WE_r^{II}(\theta)}{\partial\theta} = -n\int_{\underline{v}}^{\bar{v}}[1-H(u)]\int_0^{-\frac{\log(1-H(u))}{\theta}}\\ \times\left\{\underline{v} + (\bar{v}-\underline{v})\left(1-[1-H(u)]e^{\theta z}\right)^{\frac{1}{n-1}}\right\} ze^{-(1-\theta)z}dz\,dF(u).\end{aligned}$$

For every $u \in (\underline{v}, \bar{v})$ and every fixed $z \geq 0$,

$$\lim_{\theta \to 0^+} \left\{ \underline{v} + (\bar{v} - \underline{v}) \left(1 - [1 - H(u)] e^{\theta z}\right)^{\frac{1}{n-1}} \right\} = u.$$

Extend the integrand with respect to z to equal zero above its upper integration limit. For $\theta \in (0, 1/2)$, its absolute value is bounded by $\bar{v} z e^{-z/2}$, which is integrable on $(0, +\infty)$. The dominated convergence theorem applied with respect to z and $F(u)$ therefore gives

$$\begin{aligned}
\lim_{\theta \to 0^+} \frac{\partial WE_r^{II}(\theta)}{\partial \theta} &= -n \int_{\underline{v}}^{\bar{v}} u[1 - H(u)] dF(u) \\
&= -n \int_0^1 \left[\underline{v} + (\bar{v} - \underline{v}) x\right] (1 - x^{n-1}) dx \\
&= -\frac{n-1}{2(n+1)} \left[n\bar{v} + (n+2)\underline{v}\right],
\end{aligned} \tag{27}$$

where the second equality uses $x = F(u)$ and $H(u) = F^{n-1}(u)$.

Combining (26) and (27), we obtain

$$\begin{aligned}
\lim_{\theta \to 0^+} \frac{\partial WE^{II}(\alpha, \theta)}{\partial \theta} &= \alpha \frac{n-1}{n+1} \left[\bar{v} - (n+2)\underline{v}\right] \\
&\quad - (1 - \alpha) \frac{n-1}{2(n+1)} \left[n\bar{v} + (n+2)\underline{v}\right] \\
&= \frac{n-1}{2(n+1)} \left(\left[(n+2)\alpha - n\right]\bar{v} - (n+2)(1+\alpha)\underline{v}\right).
\end{aligned}$$

Since $\underline{v} > 0$ and $\bar{v} > \underline{v}$, the last expression is positive if and only if

$$\alpha > \frac{n\left(\frac{\bar{v}}{\underline{v}}\right) + n + 2}{(n+2)\left(\frac{\bar{v}}{\underline{v}} - 1\right)} = \frac{n}{n+2} + \frac{2(n+1)}{(n+2)\left(\frac{\bar{v}}{\underline{v}} - 1\right)}.$$

The last expression is decreasing in $\bar{v}/\underline{v}$ and increasing in n, which completes the proof. ■

Proof of Proposition 4

Proof. From Part (ii) of Lemma 3, we know that $WE^I(0, \theta) = WE_r^I(\theta) \geq WE_r^{II}(\theta) = WE^{II}(0, \theta)$ for all $\theta \in [0, 1]$, with equality if and only if $\theta = 0$. Therefore, from decomposition (15), it is sufficient to show that for any $\theta \in [0, 1]$, $WE_p^I(\theta) - WE_p^{II}(\theta) > 0$. Note that

$$
\begin{aligned}
&WE_p^I(\theta) - WE_p^{II}(\theta) \\
&= \int_{\underline{v}}^{\bar{v}} \frac{vH(v)}{(1-\theta)H(v)+\theta} dF^n(v) - \int_{\underline{v}}^{\bar{v}} \frac{v}{1-\theta}\left[1-(1-H(v))^{\frac{1-\theta}{\theta}}\right] dF_{(2)}(v) \\
&= \int_{\underline{v}}^{\bar{v}} \frac{nvF^{2n-2}(v)}{(1-\theta)F^{n-1}(v)+\theta} - \frac{v\left[1-\left(1-F^{n-1}(v)\right)^{\frac{1-\theta}{\theta}}\right]}{1-\theta} n(n-1)F^{n-2}(v)(1-F(v))dF(v) \\
&= \int_0^1 nF^{-1}(x)x^{n-2}\left[\frac{x^n}{(1-\theta)x^{n-1}+\theta} - (n-1)(1-x)\frac{1-(1-x^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}\right]dx,
\end{aligned}
$$

where the last equality results from the change of variable $x = F(v)$.

We aim to use Lemma A1 to prove $WE_p^I(\theta) - WE_p^{II}(\theta) > 0$. It is easy to prove the boundary cases when $\theta = 0$ or $\theta = 1$. Now assume $0 < \theta < 1$. Let $\widetilde{h}(x) = \frac{x^n}{(1-\theta)x^{n-1}+\theta} - (n-1)(1-x)\frac{1-(1-x^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}$ and $h(x) = x^{n-2}\widetilde{h}(x)$. We will prove in Lemma A2 below that

(i) There exists some $x_0 \in (0,1)$ such that $h(x) < 0$ (or equivalently, $\widetilde{h}(x) < 0$) for $x \in (0, x_0)$ and $h(x) > 0$ (or equivalently, $\widetilde{h}(x) > 0$) for $x \in (x_0, 1)$; and

(ii) $\int_0^1 h(x)dx = \int_0^1 x^{n-2}\widetilde{h}(x)dx > 0$.

Therefore, by Lemma A1 and Lemma A2(ii), we have that

$$
WE_p^I(\theta) - WE_p^{II}(\theta) = n\int_0^1 F^{-1}(x)h(x)dx \geq nF^{-1}(x_0)\int_0^1 h(x)dx > 0.
$$

■

Lemma A2 *For all $\theta \in (0,1)$, $n \geq 2$ and $n \in \mathbb{N}$, the following statements hold:*

(i) There exists a unique $x_0 \in (0,1)$ such that $\widetilde{h}(x) < 0$ for $x \in (0, x_0)$ and $\widetilde{h}(x) > 0$ for $x \in (x_0, 1)$; and

(ii) $\int_0^1 x^{n-2}\widetilde{h}(x)dx > 0$.

Proof. We first prove Part (i) of the lemma, which consists of two steps. First, we show that $\widetilde{h}(x) < 0$ for $x \in \left(0, \frac{n-1}{n}\right]$. Then, we prove that $\widetilde{h}(x)$ is increasing in $\left(\frac{n-1}{n}, 1\right)$. Part (i) follows immediately from these results and the fact that $\widetilde{h}(1) = 1$.

Step 1. We first prove that $\widetilde{h}(x) < 0$ for $x \in \left(0, \frac{n-1}{n}\right]$. To proceed, we first prove that for all $u, \theta \in (0,1)$, the following inequality holds:

$$\frac{1-(1-u)^{\frac{1-\theta}{\theta}}}{1-\theta} > \frac{u}{(1-\theta)u+\theta}. \tag{28}$$

In fact, this inequality is equivalent to $[(1-\theta)u+\theta](1-u)^{\frac{1-\theta}{\theta}} - \theta < 0$. Let $r(u) = [(1-\theta)u+\theta](1-u)^{\frac{1-\theta}{\theta}} - \theta$. It is easy to verify that $r'(u) < 0$ and $r(0) = 0$.

Now we start to prove that $\widetilde{h}(x) < 0$ for $x \in \left(0, \frac{n-1}{n}\right]$. Let $u = x^{n-1}$ in (28). Using inequality (28), we have that

$$\begin{aligned}
\widetilde{h}(x) &< \frac{x^n}{(1-\theta)x^{n-1}+\theta} - (n-1)(1-x)\frac{x^{n-1}}{(1-\theta)x^{n-1}+\theta} \\
&= \frac{x^{n-1}}{(1-\theta)x^{n-1}+\theta}[x-(n-1)(1-x)] \le 0
\end{aligned}$$

for $x \in \left(0, \frac{n-1}{n}\right]$, which concludes Step 1.

Step 2. We prove that $\widetilde{h}(x)$ is increasing in $\left(\frac{n-1}{n}, 1\right)$. Taking the derivative yields that

$$\begin{aligned}
\widetilde{h}'(x) &= \frac{x^{n-1}}{(1-\theta)x^{n-1}+\theta} + (n-1)\left[\frac{\theta x^{n-1}}{\left[(1-\theta)x^{n-1}+\theta\right]^2}\right. \\
&\quad \left. + \frac{1-\left(1-x^{n-1}\right)^{\frac{1-\theta}{\theta}}}{1-\theta} + \frac{(n-1)(x-1)}{\theta}x^{n-2}\left(1-x^{n-1}\right)^{\frac{1-\theta}{\theta}-1}\right] \\
&= \frac{x^{n-1}}{(1-\theta)x^{n-1}+\theta} + (n-1)q\left(y\right),
\end{aligned}$$

where $y = x^{n-1}$ and

$$q(y) := \frac{\theta y}{\left[(1-\theta)y+\theta\right]^2} + \frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{1-\theta} + \frac{(n-1)\left(1-y^{-\frac{1}{n-1}}\right)}{\theta}y(1-y)^{\frac{1-\theta}{\theta}-1}. \tag{29}$$

Therefore, to prove that $\widetilde{h}(x)$ is increasing in $\left(\frac{n-1}{n}, 1\right)$, it is sufficient to prove that $q(y) > 0$ for $y \in (d_n, 1)$ and $\theta \in (0,1)$, where $d_n := \left(\frac{n-1}{n}\right)^{n-1}$. Consider the following two cases.

Case (a): $\boldsymbol{\theta \in \left(0, \frac{1}{2}\right]}$**.** Note that when $y \in (d_n, 1)$, $1 - y^{-\frac{1}{n-1}} > 1 - d_n^{-\frac{1}{n-1}} = -\frac{1}{n-1}$. Then we have

$$q(y) > 0+\frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{1-\theta}-\frac{1}{\theta}y(1-y)^{\frac{1-\theta}{\theta}-1} = \frac{1}{1-\theta}\left[1-(1-y)^{\frac{1-\theta}{\theta}-1}\left(1+\left(\frac{1-\theta}{\theta}-1\right)y\right)\right].$$

Let $t = \frac{1-\theta}{\theta} - 1 \in [0, +\infty)$. Therefore, the inequality above is equivalent to

$$q(y) > \frac{1}{1-\theta}\left[1-(1-y)^t(1+ty)\right] := \frac{1}{1-\theta}w(y,t).$$

It is sufficient to prove that $w(y,t) \geq 0$ for $y \in (d_n, 1)$ and $t \geq 0$. First, we have that $w(y,0) = 0$ for all y. Moreover, when $t > 0$, we have that $\frac{\partial w}{\partial y} = t(t+1)y(1-y)^{t-1} > 0$, which implies that $w(y,t) > w(0,t) = 0$. Therefore, $q(y) > 0$ is proved for $\theta \in \left(0, \frac{1}{2}\right]$.

Case (b): $\boldsymbol{\theta \in \left(\frac{1}{2}, 1\right)}$**.** Recall that $y = x^{n-1}$ and $x \in \left(\frac{n-1}{n}, 1\right)$. Therefore, we have

$$(n-1)\left(1-y^{-\frac{1}{n-1}}\right) = \frac{\log y}{\log x}\left(1-\frac{1}{x}\right).$$

Note that $\left[\frac{1}{\log x}\left(1-\frac{1}{x}\right)\right]' = \frac{\log x - x + 1}{(x\log x)^2} < 0$, which implies that $\frac{1}{\log x}\left(1-\frac{1}{x}\right) < \frac{1}{\log\left(1+\frac{1}{n-1}\right)^{n-1}} := c_n$. Note that $c_n > 0$. Therefore, we have that

$$(n-1)\left(1-y^{-\frac{1}{n-1}}\right) = \frac{\log y}{\log x}\left(1-\frac{1}{x}\right) > c_n \log y. \tag{30}$$

Combining (29) and (30), we have

$$q(y) > \frac{\theta y}{\left[(1-\theta)y+\theta\right]^2} + \frac{1-(1-y)^{\frac{1-\theta}{\theta}}}{1-\theta} + \frac{c_n \log y}{\theta}y(1-y)^{\frac{1-\theta}{\theta}-1} := z(y).$$

So it is sufficient to prove that $z(y) \geq 0$ for $y \in (d_n, 1)$. We first prove that $z'(y) > 0$, then prove that $z(d_n) \geq 0$.

Taking the derivative, we have

$$\begin{aligned} z'(y) = &\frac{\theta^2 - \theta(1-\theta)y}{\left[(1-\theta)y+\theta\right]^3} + \frac{1}{\theta}(1-y)^{\frac{1-\theta}{\theta}-1} + \frac{c_n(\log y + 1)}{\theta}(1-y)^{\frac{1-\theta}{\theta}-1} \\ &- \left(2-\frac{1}{\theta}\right)\frac{c_n \log(1/y)}{\theta}y(1-y)^{\frac{1-\theta}{\theta}-2} \end{aligned}$$

$$
\begin{aligned}
&> \frac{\theta^2-\theta(1-\theta)y}{\left[(1-\theta)y+\theta\right]^3}+\frac{1}{\theta}(1-y)^{\frac{1-\theta}{\theta}-1}+\frac{c_n(\log y+1)}{\theta}(1-y)^{\frac{1-\theta}{\theta}-1}-\frac{c_n\log(1/y)}{\theta}y(1-y)^{\frac{1-\theta}{\theta}-2}\\
&= \frac{\theta^2-\theta(1-\theta)y}{\left[(1-\theta)y+\theta\right]^3}+\frac{c_n(1-y)^{\frac{1-\theta}{\theta}-2}}{\theta}\left[\log y+\left(\frac{1}{c_n}+1\right)(1-y)\right],
\end{aligned}
$$

where the inequality follows from $\theta \in \left(\frac{1}{2},1\right)$. We have that $\theta^2-\theta(1-\theta)y > 0$ because $\theta \in \left(\frac{1}{2},1\right)$. To prove that $z'(y) > 0$, it is sufficient to prove that $\log y + \left(\frac{1}{c_n}+1\right)(1-y) \geq 0$ for $y \in (d_n,1)$.

Let $\kappa(y) := \log y + \left(\frac{1}{c_n}+1\right)(1-y)$. Note that $\kappa'(y) > 0$ if $y \in \left(d_n, \left(\frac{1}{c_n}+1\right)^{-1}\right)$ and $\kappa'(y) < 0$ if $y \in \left(\left(\frac{1}{c_n}+1\right)^{-1}, 1\right)$. Moreover, note that $\kappa(1) = 0$. Therefore, it is sufficient to prove that $\kappa(d_n) \geq 0$. In fact, note that $c_n \log d_n = -1$. Therefore, $\kappa(d_n) = \log d_n + \left(\frac{1}{c_n}+1\right)(1-d_n) = d_n \log d_n - d_n + 1 > 0$ for $d_n = \left(\frac{n-1}{n}\right)^{n-1} \in (0,1)$. Thus, we have proved that $z'(y) > 0$ for $y \in (d_n,1)$.

It remains to prove that $z(d_n) \geq 0$. In fact, we have

$$
\begin{aligned}
z(d_n) &= \frac{\theta d_n}{[(1-\theta)d_n+\theta]^2}+\frac{1-(1-d_n)^{\frac{1-\theta}{\theta}}}{1-\theta}+\frac{c_n\log d_n}{\theta}d_n(1-d_n)^{\frac{1-\theta}{\theta}-1}\\
&= \frac{\theta d_n}{[(1-\theta)d_n+\theta]^2}+\frac{1-(1-d_n)^{\frac{1-\theta}{\theta}}}{1-\theta}-\frac{1}{\theta}d_n(1-d_n)^{\frac{1-\theta}{\theta}-1}\\
&\geq \frac{\theta d_n}{[(1-\theta)d_n+\theta]^2}+\frac{d_n}{\theta}-\frac{d_n}{\theta(1-d_n)}\left(1-\frac{1-\theta}{\theta}d_n\right)\\
&= \frac{d_n}{\theta}\left[\frac{\theta^2}{[(1-\theta)d_n+\theta]^2}-\left(2-\frac{1}{\theta}\right)\frac{d_n}{1-d_n}\right],
\end{aligned}
$$

where the second line follows from $c_n \log d_n = -1$ and the third line follows from Bernoulli's inequality $(1-d_n)^{\frac{1-\theta}{\theta}} < 1 - \frac{1-\theta}{\theta}d_n$. It is sufficient to prove that

$$
\frac{\theta^2}{[(1-\theta)d_n+\theta]^2}-\left(2-\frac{1}{\theta}\right)\frac{d_n}{1-d_n} \geq 0, \tag{31}
$$

for $d_n = \left(\frac{n-1}{n}\right)^{n-1}$, $n \geq 2$, and $\theta \in \left(\frac{1}{2},1\right)$. Note that the left-hand side of (31) is decreasing in d_n. Moreover, $d_n = \left(\frac{n-1}{n}\right)^{n-1}$ is decreasing in n. It remains to prove that (31) holds for $\max_n d_n = \frac{1}{2}$ when $n = 2$, which is equivalent to

$$\frac{(1-\theta)^2(1+2\theta)}{\theta(1+\theta)^2} \geq 0,$$

which obviously holds for $\theta \in \left(\frac{1}{2}, 1\right)$.

Next, we prove Part (ii) of the lemma. The statement to be proved is equivalent to

$$\int_0^1 \frac{x^{2n-2}}{(1-\theta)x^{n-1}+\theta}dx > \int_0^1 x^{n-2}(n-1)(1-x)\frac{1-(1-x^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}dx. \tag{32}$$

The right-hand side of (32) can be expressed as

$$\begin{aligned}
\int_0^1 x^{n-2}(n-1)(1-x)\frac{1-(1-x^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}dx &= \int_0^1 y^{n-2}(n-1)(1-y)\frac{1-(1-y^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}dy \\
&= \int_0^1 \frac{1-(1-y^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}(n-1)y^{n-2}dy \int_y^1 dx \\
&= \int_0^1 dx \int_0^x \frac{1-(1-y^{n-1})^{\frac{1-\theta}{\theta}}}{1-\theta}d(y^{n-1}) \\
&= \int_0^1 \frac{x^{n-1}+\theta\left(1-x^{n-1}\right)^{\frac{1}{\theta}}-\theta}{1-\theta}dx.
\end{aligned}$$

Therefore, it is sufficient to prove that for all $x \in (0,1)$,

$$\frac{x^{2n-2}}{(1-\theta)x^{n-1}+\theta} > \frac{x^{n-1}+\theta\left(1-x^{n-1}\right)^{\frac{1}{\theta}}-\theta}{1-\theta}.$$

Let $y = x^{n-1}$. The inequality is equivalent to

$$\begin{aligned}
&\frac{y}{1-\theta} + \left(\frac{\theta}{1-\theta}\right)^2 \times \frac{1}{(1-\theta)y+\theta} - \frac{\theta}{(1-\theta)^2} > \frac{y+\theta(1-y)^{\frac{1}{\theta}}-\theta}{1-\theta} \\
\Leftrightarrow &\left(\frac{\theta}{1-\theta}\right)^2 \times \frac{1}{(1-\theta)y+\theta} - \left[\frac{\theta}{(1-\theta)}\right]^2 > \frac{\theta}{1-\theta}(1-y)^{\frac{1}{\theta}} \\
\Leftrightarrow &(1-y)^{-\frac{1-\theta}{\theta}} > 1 + \frac{1-\theta}{\theta}y,
\end{aligned}$$

which is Bernoulli's inequality. This concludes the proof. ■

Appendix B Additional Numerical Results

Figure B1 extends the uniform exercise in the main text to $n = 3$ and $n = 4$. Projection bias continues to strengthen the effect of θ on total effort under the second-price rule, but the amplification becomes smaller as the number of contestants increases. The projection-induced region over which θ raises the winner's effort also contracts rapidly toward $\theta = 0$.

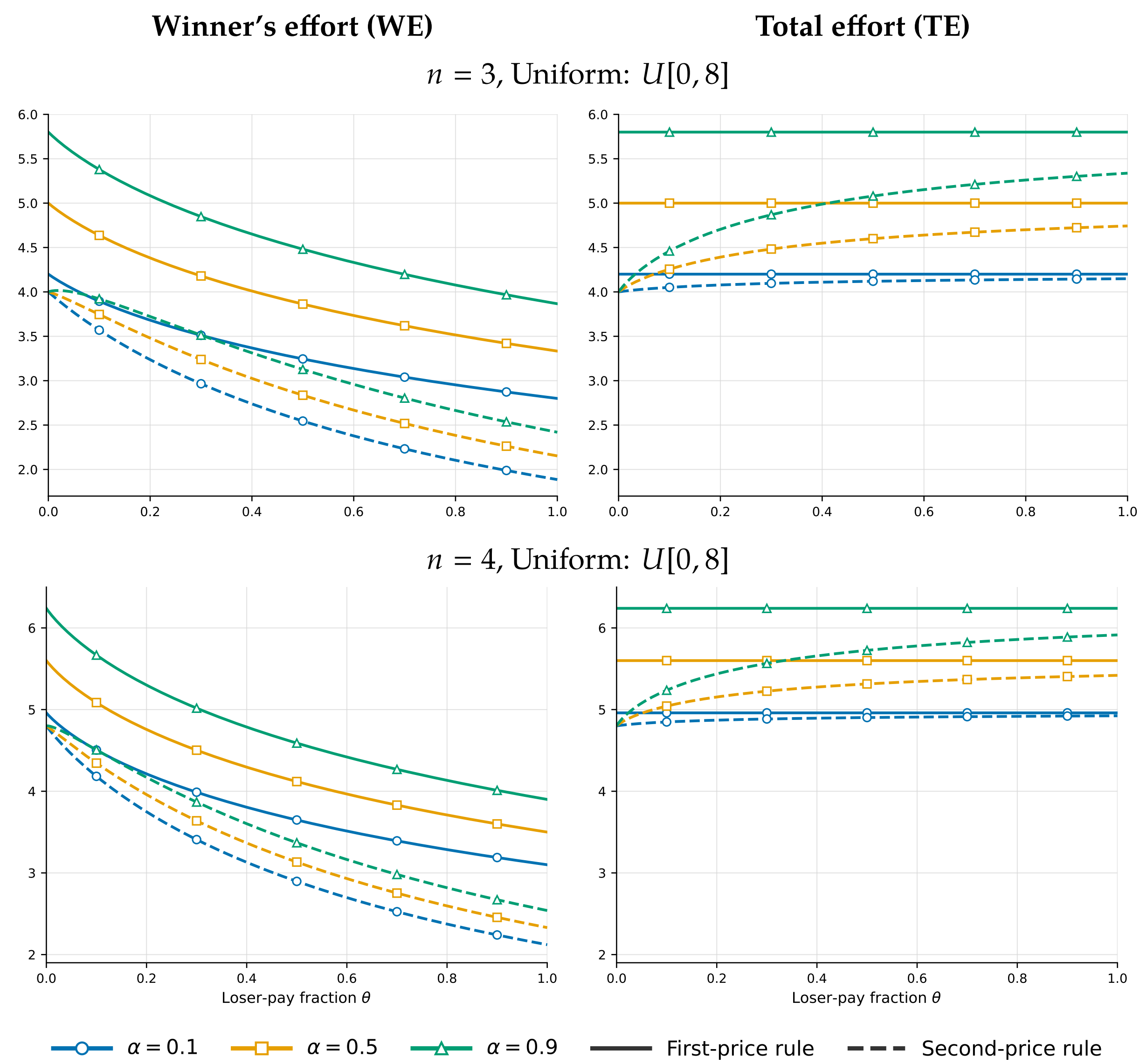

Figure B1: Expected winner's effort (left) and total effort (right) under the uniform distribution for three and four contestants. Circles, squares, and triangles indicate $\alpha = 0.1, 0.5,$ and 0.9, respectively; solid and dashed curves represent the first- and second-price rules.