On Rent Dissipation in Dynamic Multi-battle Contests

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Introduction and Motivation

- Competitions often unfold over multiple phases.
  - Contenders confront each other repeatedly in a sequence of battles.
  - Final victory requires accumulating sufficiently many intermediate successess.
  - Examples: Military warfare; U.S. presidential primaries; high-profile patent litigations (e.g., litigation between Apple and Samsung from 2011 to 2018).
- A substantial body of research on dynamic multi-battle contests (see, e.g., Harris and Vickers, 1987; Klumpp and Polborn, 2006; Konrad and Kovenock, 2009; Fu, Lu, and Pan, 2015).
  - Contenders' dynamic incentives.
  - How the structure of the contest shapes effort provision.

Introduction and Motivation

- We examine rent dissipation, or equivalently, effort provision, in a general dynamic contest.
- The paper answers the following **two open questions**:
  - **Can (almost) full rent dissipation happen in dynamic contests?**
  - **If so, for what kinds of contests?**

Model Setup

Component Battles

- Two risk-neutral players A and B compete head-to-head for a prize of common value $v > 0$.
- The contest consists of a sequence of successive *component battles*, and the final winner is determined by the history of battle outcomes.
- In each component battle, player $\ell \in \{A, B\}$ simultaneously exerts an effort $x_\ell \in \mathbb{R}_+ := [0, +\infty)$, incurring a unity marginal cost.
- For a given effort profile (x_A, x_B), player A wins the current battle with a probability $p(x_A, x_B) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, 1]$, and player B wins with complementary probability $p(x_B, x_A) = 1 - p(x_A, x_B)$.
- $p(x_A, x_B)$: The *success function* (SF) for component battles.

Model Setup
Component Battles

- Baseline model: Homogeneous (of degree zero) success function
  - For given (x, x'), a player, by exerting an effort x, wins the battle with a probability

$$p(x, x') = \gamma\left(\frac{x}{x'}\right),$$

where $\gamma : [0, +\infty] \longrightarrow [0, 1]$ is a continuous and twice-differentiable function with $\gamma(0) = 0$, $\gamma(+\infty) = 1$, $\gamma' > 0$, $\gamma'' \leq 0$, and $\gamma(x) + \gamma(1/x) = 1$.
  - E.g., Tullock Success Function:

$$p(x, x') = \begin{cases} \dfrac{x^r}{x^r + (x')^r}, & \text{if } (x, x') \neq (0, 0), \\ 1/2, & \text{if } (x, x') = (0, 0), \end{cases}$$

with $r \in (0, 1]$.

Model Setup

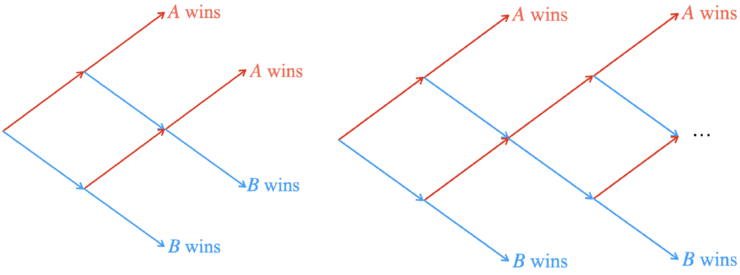
Contest Architecture

- At the beginning of each battle, the outcomes of all previous battles are commonly known.
- An *outcome path* is denoted by $\ell^t := (\ell_s)_{s=1}^t$ for $t > 1$, where each element $\ell_s \in \{A, B\}$ indicates the winner of battle s.
- *Terminal histories*: An outcome path ending with the battle that determines the ultimate winner of the contest according to the contest rule.
- *History* of the contest: An outcome path that does not extend beyond any terminal history.

Model Setup

Contest Architecture

- Terminal histories in best-of-3 bontest: $\{AA, BB, ABA, BAB, ABB, BAA\}$
- Terminal histories in tug-of-war with margin 2: $\{AA, BB, ABBB, BAAA, \ldots\}$



(a) Best-of-3 Contest. (b) Tug-of-war with Margin 2.

Model Setup

Benchmark: Bounded Rent Dissipation in Contests

- We consider general dynamic contests rather than focusing on a particular architecture.
- Let $L(\mathcal{M}) := \min\{t : \ell^t \in H^\dagger\}$ denote the *minimum length* of the contest $\mathcal{M}$, i.e., the length of the shortest terminal history.
- Remark: For a contest to be well-defined, we must have $L(\mathcal{M}) < +\infty$.

Model Setup

Benchmark: Bounded Rent Dissipation in Contests

Theorem 1 (An Upper Bound on Rent Dissipation)

The expected total effort in any equilibrium (if one exists) of a contest $\mathcal{M}$ *is less than* $\left\{1 - [\phi(1)]^{L(\mathcal{M})}\right\} v$*. Therefore, rent does not fully dissipate in any contests.*

- $\phi(1) \in (0, 1/2)$ is a constant that is pinned down by the battle success function.
- The upper bound of rent dissipation is determined by the **shortest** terminal history.
- Remark: **Even infinite-horizon contests cannot fully dissipate rent**.
  - Example: A tug-of-war contest with a margin $N < +\infty$: $\left\{1 - [\phi(1)]^{N}\right\} v$.

Question

- Consider a sequence of contests $\{\mathcal{M}_k\}_{k\in\mathbb{N}_{++}}$ where each $\mathcal{M}_k$ is a contest.
- Let $L(\mathcal{M}_k) \longrightarrow +\infty$ as $k \longrightarrow +\infty$, and consider the limiting property of this sequence.
- Can the sequence of expected equilibrium total efforts approach v?

Case of Tug-of-War Contests

- Consider a tug-of-war contest with margin $N \geq 1$, whereby a player must win N more battles than his opponent to secure the final victory.
  - Adopt symmetric Markov Perfect Equilibrium (MPE) as the solution concept of the game.
  - The state of a tug–of-war contest can be described by i over $\{i \in \mathbb{N} | -N \leq i \leq N\}$, whereby a player who leads by i battles.

Proposition 1 (Existence of Symmetric MPE in Tug-of-War)

A unique symmetric Markov perfect equilibrium (MPE) exists in a tug-of-war with margin $N \geq 1$.

Case of Tug-of-War Contests

- Recall: $V(0; N, v)$ represents a player's expected continuation value for the contest.

Theorem 2 (Bounded Rent Dissipation in Tug-of-War)

There exists a constant $\alpha > 0$ such that $V(0; N, v) \geq \alpha v$ for all $N \geq 1$ and $v > 0$. Consequently, the expected total effort in any symmetric MPE of the tug-of-war with any margin $N \geq 1$ is bounded from above by $(1 - 2\alpha)v$.

- Rent cannot fully dissipate in a tug-of-war contest, **even when the margin requirement for victory grows unboundedly.**

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- Rent cannot fully dissipate in a tug-of-war contest, **even when the margin requirement for victory grows unboundedly.**
- Why?

Case of Tug-of-War Contests

- Let $Q(i; N, v)$ denote the probability that a player who leads by i battles wins the entire contest.

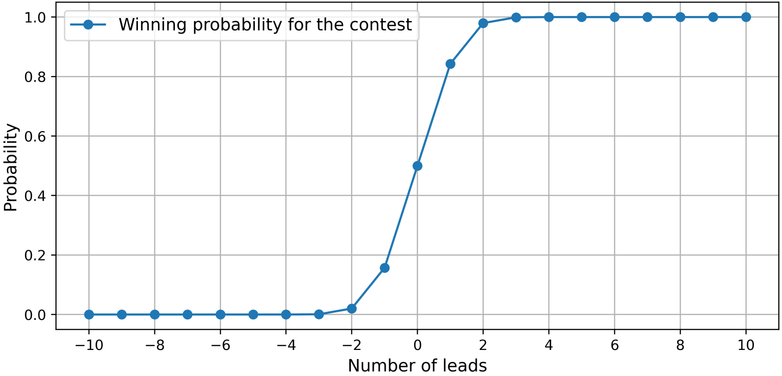
Proposition 2 (Discouragement Effect)

In the symmetric MPE of the tug-of-war with margin N, it holds that $\lim_{i \to +\infty} \inf_{N>i} Q(i; N, v) = 1$; that is, the frontrunner secures an almost-sure win once its lead is sufficiently large.

- Discouragement Effect: Although $N > i$, as the lead i is sufficiently large, the final outcome of the contest boils down to certainty.

Role of the Discouragement Effect

- Example: Tullock success function, $r = 0.75$, tug-of-war with margin 11.
- Strong Start: A 2-win lead practically guarantees final victory, despite needing 9 more.



Exchangeability and Discouragement Effect

Definition (Exchangeable Contests)

A contest is *exchangeable* if any two histories that differ only in the order of the battle outcomes lead to the same subgame or final outcome.

- Exchangeability is satisfied by many dynamic contests.
  - Examples: Tug-of-war contests and best-of-$(2K+1)$ with $K \geq 1$.
- Remark: For an exchangeable contest, a history can be summarized by the *number of battle wins* each player has.

Exchangeability and Discouragement Effect

- Exchangeability allows a frontrunner to *accumulate advantage*.
  - For player A, given past outcomes $h = (\ell_1, \dots, \ell_t)$, his value of winning the next battle is given by $\Delta_A(\ell_1, \dots, \ell_t) = V(\ell_1, \dots, \ell_t, A) - V(\ell_1, \dots, \ell_t, B)$.
  - With exchangeability, the outcome path of $(\ell_1, \dots, \ell_t, A)$ is the same as the alternative history $(A, \ell_1, \dots, \ell_t)$, and so is $(\ell_1, \dots, \ell_t, B)$ as $(B, \ell_1, \dots, \ell_t)$.
- Exchangeability enables a *long memory*.
  - The impact of an early outcome lasts even after a large number of subsequent battles.
- We introduce a framework for studying the limit of a sequence of exchangeable contests in the paper, but I'll skip the nitty-gritty here.

Details.

Exchangeable Extensions

- Fix an initial symmetric exchangeable contest $\mathcal{M}_1$ and a sequence of extensions $\{\mathcal{M}_k\}_{k=2}^{+\infty}$.
- Let $V_{0;k}$ denote a player's equilibrium continuation payoff at the beginning of contest $\mathcal{M}_k$.

Theorem 3 (Limit of Exchangeable Contests)

Under homogeneous success function, for any sequence of exchangeable extensions $\{\mathcal{M}_k\}_{k=1}^{+\infty}$ such that $\lim_{k\to+\infty} V_{0;k}$ exists, we have

$$\lim_{k\to+\infty} V_{0;k} \geq \widetilde{\alpha} v$$

for some constant $\widetilde{\alpha} > 0$ that depends only on the success function.

$\Longrightarrow$ Full rent dissipation requires a nonexchangeable contest.

Consecutive-win Contests: Beyond Exchangeability

- A K-consecutive-win contest, with $K \in \mathbb{N}_{++}$: The player who first wins K battles in a row is awarded the final prize.
- Remark: This contest is not exchangeable.
  - Set $K = 4$; (A, A, B) and (A, B, A) do not lead to the same subsequent subgame: no long-memory.

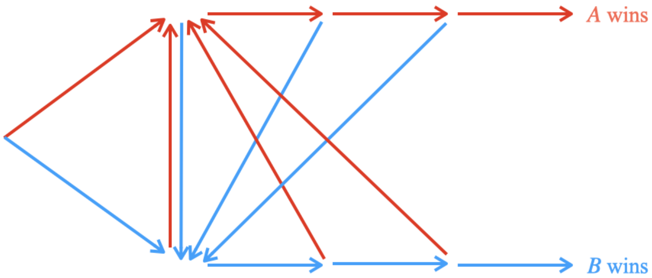


Figure: Illustration of 4-consecutive-win contest.

Consecutive-win Contests: Beyond Exchangeability

- State space $\{i \in \mathbb{N} : -K \leq i \leq K\}$:
  - if $i > 0$, the player currently holds a winning streak of length i;
  - if $i < 0$, he currently holds a losing streak of length $-i$;
  - $i = 0$ represents the start of the game.
- We show that a consecutive-win contest yields a unique symmetric MPE.
- Let $\widehat{Q}(i; K, v)$ denote the probability that a player in state i eventually wins the contest.

Proposition 3 (Bounded Accumulated Advantage in Consecutive-win Contests)

In the symmetric MPE of the consecutive-win contest, it holds that $\lim_{K>|i|, K\to+\infty} \widehat{Q}(i; K, v) = 1/2$ *for all* $i \in \mathbb{N}$.

- Absence of Discouragement Effect: With a large gap, players' winning odds remain even.

Consecutive-win Contests: Beyond Exchangeability

Theorem 4 (Full Rent Dissipation in the Limit of Consecutive-win Contests)

For all $\epsilon > 0$, there exists $K^{\dagger}$ such that $\widehat{V}(0; K, v) < \epsilon v$ in the equilibrium of the K-consecutive-win contest for $K > K^{\dagger}$, which implies that the expected total effort is greater than $(1 - 2\epsilon)v$.

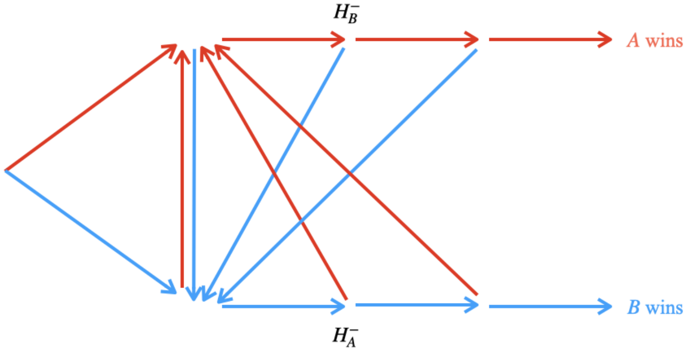
- The K-consecutive-win contest—which violates exchangeability—fully dissipates the rent in the limit.

Lesson from Consecutive-win

- Why do consecutive-win contests fully dissipate rent in the limit?
- Intuition: A player may become strong, but the advantage must not persist.
- **Without exchangeability:** Advantage builds *gradually*, but leadership can be lost *abruptly*.
  - For the leader, a battle loss does not just remove the advantage accumulated from **one** previous battle win, it more than offsets **all** accumulated battle wins.
  - This is in sharp contrast with exchangeable contests.
- This keeps the laggard hopeful and points to a general sufficient condition.
- Next, we formalize the idea that **dominance is transient**.

Lesson from Consecutive-win

- In a K-consecutive-win contests, for $K^\dagger \in \{1, \ldots, K\}$, let $H_\ell^-(K^\dagger)$ denote the set of histories in which player $\ell \in \{A, B\}$ currently has a losing streak of length $K^\dagger$.
- $H_A^-(2) = \{BB, ABB, AABB, BABB\}$, $H_B^-(2) = \{AA, BAA, BBAA, ABAA\}$.



Lesson from Consecutive-win

- Dominance is transient in (the limit of) consecutive-win contests in the following sense.

Proposition 4 (Dominance Is Transient in Consecutive-win Contests)

Fix any small $\epsilon > 0$. There exist integers $K^{\dagger} > 0$ and $K' > K^{\dagger}$ such that, for all K-consecutive-win contests with $K > K'$, the following holds:

(i) *For every $h \in H_{\ell}^{-}(K^{\dagger})$, player ℓ's continuation value is small: $V_{\ell}^{-}(h) < \epsilon v$.*

(ii) *Along the equilibrium path, the probability that the realized history passes through both $H_{A}^{-}(K^{\dagger})$ and $H_{B}^{-}(K^{\dagger})$ exceeds $1 - \epsilon$.*

- Remark: For a general contest $\mathcal{M}$, we can define transient dominance property with appropriately chosen (H_{A}^{-}, H_{B}^{-}).

Transient Dominance

Definition (Transient Dominance in a General Setting)

Consider a contest $\mathcal{M}$ and its equilibrium. Fix any small $\epsilon > 0$. We say that the equilibrium has the *transient dominance property* if the following conditions are satisfied:

(i) There exist two subsets of the nonterminal histories, H_A^- and H_B^-, such that at any history $h \in H_\ell^-$, the continuation value $V_\ell(h) \leq \epsilon v$.

(ii) In equilibrium, the probability that the realized history passes through both H_A^- and H_B^- is at least $1 - \epsilon$.

- Condition (i) formalizes *dominance*: For each player ℓ, it identifies a set H_ℓ^- of nonterminal histories where player ℓ is in a "weak" position with a small continuation value.
- Condition (ii) formalizes *transience*. In particular, the probability of bouncing back to H_B^- is large conditional on the history reaching H_A^-, and vice versa.

Transient Dominance

Theorem 5 (Transient Dominance Guarantees Almost Full Rent Dissipation)

Fix any $\epsilon > 0$. The expected equilibrium total effort of a contest $\mathcal{M}$ with prize v exceeds $(1 - 4\epsilon)v$ if its equilibrium exhibits the transient dominance property.

- The transient dominance property leads to almost-full rent dissipation in the contest.
  - This property is not only sufficient but also necessary for almost full rent dissipation.
- Transient dominance clearly defies the long-memory property enabled by exchangeability.

Transient Dominance: Iterated Incumbency Contests

- We model protracted competition in uncertain environments as **iterated incumbency contests**.
  - Two firms compete in a market of evolving technologiesprogress and shifting consumer tastes.
  - A firm's temporary market dominance may not suffice to eliminate its opponent or to establish permanent market leadership.
  - When the underlying fundamentals change, the laggard may regain the opportunity to overtake the frontrunner.
- Schumpeterian competition: Radical innovations and technological paradigm shifts often function as de facto reset mechanisms.
  - Nokia's dominance in feature phones, Kodak's leadership in film photography, and Blockbuster's command of physical video rental
- Leadership is inherently contingent and transient: The dominace of incumbency does not accumulate indefinitely across regimes. $\Longrightarrow$ Full rent dissipation over a long horizon.

Transient Dominance: Iterated Incumbency Contests

Theorem 7 (Almost Full Dissipation in Iterated Incumbency Contests)

Consider an N-round iterated incumbency contest with subcontest $\mathcal{M}^{sub}$. If the subcontest is sufficiently biased, the iterated incumbency contest satisfies the transient dominance property as $N \longrightarrow +\infty$, which leads to almost full rent dissipation.

Concluding Remarks

- This paper provides a general analysis of dynamic multi-battle contests to explore how contest architecture governs rent dissipation.
- Any contest cannot fully dissipate its rent, despite the possibility of an infinite horizon.
- **Exchangeability** prevents full rent dissipation even asymptotically.
- A **transient dominance** property defies the long memory caused by exchangeability and ensures almost full rent dissipation.

Appendix

(Exchangeability-preserving) Extensions

Definition (Exchangeability-preserving Extended Contests)

For a given symmetric exchangeable contest $\mathcal{M}$, a symmetric contest $\mathcal{M}'$ is said to be the N-*extension* of $\mathcal{M}$, with $N \geq 2$, if both the following conditions are met:

(i) $\mathcal{M}'$ is exchangeable, and the subgames following both histories (A, B) and (B, A) are $\mathcal{M}$;

(ii) from the start of $\mathcal{M}'$, a player wins the contest after winning exactly N battles in a row.

- Examples:
  - The best-of-$(2K + 1)$ contest is the $(K + 1)$-extension of the best-of-$(2K - 1)$ contest.
  - A tug-of-war with margin-N is the N-extension of itself.

(Exchangeability-preserving) Extensions

- To illustrate this definition, we consider a best-of-3 contest.

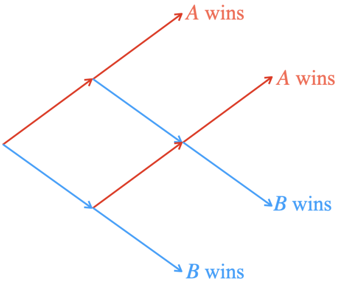


Figure: Best-of-3 (i.e., win-with-2) contest.

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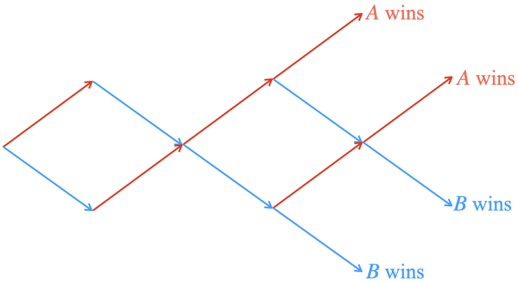

Figure: An exchangeable extension “prepends” battles to the original contest.

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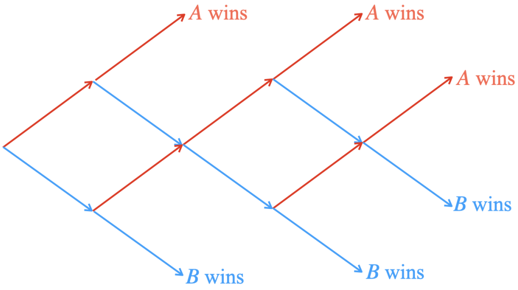

Figure: 2-extension of best-of-3.

(Exchangeability-preserving) Extensions

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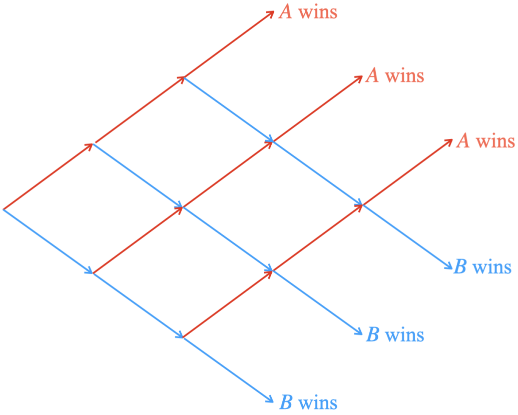

Figure: 3-extension of best-of-3.

Exchangeable Extensions

- With this definition, we proceed to study the limit of a sequence of exchangeable contests $\{\mathcal{M}_k\}_{k=1}^{+\infty}$, where $\mathcal{M}_{k+1}$ is the N_{k+1}-extension of $\mathcal{M}_k$.
- If $\lim_{k\to+\infty} N_k < +\infty$, since $L(\mathcal{M}_{k+1}) \leq N_{k+1}$ by definition, Theorem 1 implies that rent cannot fully dissipate in the limit.
- So we focus on the $\lim_{k\to+\infty} N_k = +\infty$ case.